Integration

Table of standard integrals

Standard Integrals	
$f(x)$	$\int f(x)dx$
$\sin ax$	$-\frac{1}{a}\cos x + C$
$\cos ax$	$\frac{1}{a}\sin ax + C$
$\sec^2 ax$	$\frac{1}{a}\tan ax + C$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{x}$	$\ln x + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$

You are expected to already know that

$f(x)$	$F(x)$
$\sin ax$	$-\frac{1}{\mathrm{a}}\cos x + C$
$\cos ax$	$\frac{1}{a}\sin x + C$

Lesson 1 – Standard Integrals

For integration to proceed, functions must be expressed in **integrable form**.

So always expand brackets or simplify fractions before integrating

Example 1 $\int x(x^2+3)\,dx = \int x^3 + 3x \; dx = \frac{1}{4}x^4 + \frac{3}{2}x^2 + C$

Example 2 $\int \frac{x^2+1}{x}dx = \int \frac{x^2}{x} + \frac{1}{x} \; dx = \int x + \frac{1}{x} \; dx = \frac{1}{2}x^2 + \ln x + C$

When integrating composite functions $h(x) = f(g(x))$ where the inner function takes the form $g(x) = ax + b$

The result of integration is $H(x) = \frac{F(g(x))}{g'(x)}$

(integrate the outer function and divide by the derivative of the inner function)

Example 3 $\int \sqrt{5x-1} \; dx = \int (5x-1)^{\frac{1}{2}} \; dx = \frac{(5x-1)^{\frac{3}{2}}}{\frac{3}{2}\times 5} + C = \frac{2}{15}(5x-1)^{\frac{3}{2}} + C$

Example 4 $\int \sec^2(3x+1)\,dx = \frac{1}{3}\tan(3x+1) + C$

Example 5 $\int 2e^x \, dx = 2\int e^x = 2e^x + C$ **but** $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$,

Example 6 $\int \frac{2}{x} \, dx = 2\int \frac{1}{x} \, dx = 2\ln x + C$ **but** $\int \frac{1}{2x+5} \, dx = \frac{1}{2}\ln(2x+5) + C$

Example 7 $\int_1^{16} \left(\sqrt[4]{\frac{1}{x}}\right) dx = \left[\frac{x^{\frac{3}{4}}}{\frac{3}{4}}\right]_1^{16} = \frac{4}{3}\left[x^{\frac{3}{4}}\right]_1^{16} = \frac{4}{3}\left(16^{\frac{3}{4}} - 1^{\frac{3}{4}}\right) = \frac{4}{3}(8-1) = \frac{28}{3}$

For more complex composite functions such as $\int (x^3+1)^3 \; dx$

It is better to expand the expression using the binomial theorem

$$\int (x^3+1)^3 \; dx = \int x^9 + 3x^6 + 3x^3 + 1 \; dx = \frac{1}{10}x^{10} + \frac{3}{7}x^7 + \frac{3}{4}x^4 + x + C$$

Remember, you can check your integration by differentiating the answer.

Or use **https://www.integral-calculator.com/**

The integral-calculator also allows you to evaluate definite integrals. Just use Options to enter the upper and lower bounds for the integration.

In MIA textbook - Exercise 7.1 Q1 – 3, questions 4 and 5 for extension
In Leckie and Leckie - Exercise 3A and Ex 3C (logs and exponentials)

Lesson 2 – Integration by substitution

This where you perform an apparently difficult piece of integration by making a simple substitution.   This works well with functions in the form $g'(x) \times f\big(g(x)\big)$ since

$$\int f\big(g(x)\big) \times g'(x)dx = \ F(u) + C \ = \ F\big(g(x)\big) + C$$

This is an easy process as long as you correctly identify the inner function (the **subject for change**) which you will both differentiate and substitute into the integral;

To find the integral of	$\int x(x^2+5)^3 dx$
1. Identify the inner function (subject for change)	$u = x^2 + 5$
2. Differentiate and rearrange if necessary	$du = 2x\ dx$ $\frac{1}{2}du = x\ dx$
3. In the integral, replace $g(x)$ with u and $g'(x)$ with the LHS of the derivative	$\int x(x^2+5)^3 dx$ $\int \frac{1}{2}(u)^3 du$ $\frac{1}{2}\int (u)^3\, du$
4. integrate with respect to u	$\frac{1}{2} \times \frac{1}{4}u^4 + C$
5. Substitute for u and simplify	$\frac{1}{8}(x^2+5)^4 + C$
The final answer can always cross checked through differentiation $\frac{d}{dx}\left(\frac{1}{8}(x^2+5)^4 + C\right) = \frac{4}{8}(x^2+5)^3 \times 2x = x(x^2+5)^3$	
The subject for change is almost always specified in exam questions:	

Example 2

$\int \frac{1}{1-2x} dx$, where $u = 1 - 2x$, $du = -2\,dx$, $-\frac{1}{2} du = 1dx$

$$\int \frac{1}{1-2x} dx \to \int \frac{1}{u} \times -\frac{1}{2} du = -\frac{1}{2}\int \frac{1}{u} du = -\frac{1}{2}\ln u + C = -\frac{\mathbf{1}}{\mathbf{2}}\mathbf{\ln\,(1-2x)} + \boldsymbol{C}$$

Example 3

$\int e^{\sin x} \cos x \; dx$, $u = \sin x$, $du = \cos x\, dx$

$$\int e^{\sin x} \cos x \; dx \to \int e^{u}\, du = e^{u} + C = \boldsymbol{e^{sinx}} + \boldsymbol{c}$$

Example 4

$\int x^4(1+x^5)^3 dx$, $u = 1 + x^5$, $du = 5x^4\,dx$, $\frac{1}{5} du = x^4 dx$

$$\int x^4(1+x^5)^3 dx \to \int u^3 \times \frac{1}{5} du = \frac{1}{5}\int u^3\, du = \frac{1}{5} \times \frac{1}{4} u^4 + c$$

$$= \frac{\mathbf{1}}{\mathbf{20}}\left(\mathbf{1} + \boldsymbol{x^5}\right)^{\mathbf{4}} + \boldsymbol{C}$$

Example 5

$\int 5x\sqrt{2x^2+7}\; dx$, $u = 2x^2 + 7$, $du = 4x\,dx$, $\frac{5}{4} du = 5x^2 dx$

$$\int 5x\sqrt{2x^2+7}\; dx \to \frac{5}{4}\int u^{\frac{1}{2}}\, du = \frac{5}{4} \times \frac{2}{3} u^{\frac{3}{2}} = \frac{\mathbf{5}}{\mathbf{6}}(\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{7})^{\frac{\mathbf{3}}{\mathbf{2}}} + \boldsymbol{C}$$

In MIA textbook - Exercise 7.2 All of question 1, one of Q2 to 4
In Leckie and Leckie - Exercise 3G Q1,4,5,6.

Lesson 3 – Evaluate a definite integral using substitution

$$\int_a^b f(g(x)) \times g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du = [F(u)]_{g(a)}^{g(b)}$$

Evaluate $\int_1^3 (x-1)(x^2-2x)^3 dx$	
1. Identify the inner function	$u = x^2 - 2x$
2. Differentiate and rearrange if necessary	$du = 2x - 2\,dx$ $\frac{1}{2}du = x - 1\,dx$
3. Evaluate the new limits $g(b)$ and $g(a)$	$x = 3, \quad u = 3$ $x = 1, \quad u = -1$
4. In the original integral, make substitutions for $g(x)$, $g'(x)$ and the limits a and b	$\frac{1}{2}\int_{-1}^{3} (u)^3 du$
5. integrate with respect to u,	$\frac{1}{2}\left[\frac{1}{4}u^4\right]_{-1}^{3}$
6. Substitute for u and evaluate	$\frac{1}{8}[81 - 1] = 10$

Example 2 Evaluate $\int_0^{\sqrt{5}} 6x\sqrt{4+x^2}\,dx$

let $u = 4 + x^2$

$du = 2x\ dx$,

$3du = 6x\,dx$

If $x = \sqrt{5},\ u = 4 + 5 = 9$

$x = 0, \quad u = 4 + 0 = 4$

$$\int_0^{\sqrt{5}} 6x\sqrt{4+x^2}\,dx \to \int_4^9 3\sqrt{u}\,du = \left[2u^{3/2}\right]_4^9 = [2 \times 27 - 2 \times 8] = 38$$

Example 3 Use the substitution $u = 2 + x$ to evaluate $\int_0^1 x(2+x)^3\ dx$

let $u = 2 + x$ and $\boldsymbol{u - 2 = x}$ If $x = 1,\ u = 2 + 1 = 3$

then $du = 1 \times dx$, $x = 0,\ u = 2 + 0 = 2$

$$\int_0^1 x(2+x)^3\ dx \rightarrow \int_2^3 (u-2)u^3\ du = \int_2^3 u^4 + 2u^3\ du$$

$$\left[\frac{1}{5}u^5 - \frac{1}{2}u^4\right]_2^3 = \left[\left(\frac{1}{5} \times 3^5 - \frac{1}{2} \times 3^4\right) - \left(\frac{1}{5} \times 2^5 - \frac{1}{2} \times 2^4\right)\right] = \frac{97}{10}$$

Example 4 Evaluate $\int_2^3 \frac{x+1}{x^2+2x-6}\ dx$

let $u = x^2 + 2x - 6$

$du = 2x + 2\ dx$, If $x = 3,\ u = 9 + 6 - 6 = 9$

$\frac{1}{2}du = x + 1\ dx$ $x = 2,\ u = 4 + 4 - 6 = 2$

$$\int_3^3 \frac{x+1}{x^2+2x-6}\ dx \rightarrow \frac{1}{2}\int_2^9 \frac{1}{u}\ du = \frac{1}{2}\int_2^9 \frac{1}{u}\ du = \frac{1}{2}[\ln u\ ]_2^9 = \frac{1}{2}\ln\left(\frac{9}{2}\right)$$

Example 5 Use the substitution $u = \cos x$ to evaluate $\int_{\pi/6}^{\pi/4} \frac{\sin x}{\cos^2 x}\ dx$

let $u = \cos x$

$du = -\sin x\ dx$, If $x = \pi/4,\ u = 1/\sqrt{2}$

$-du = \sin x\ dx$ $x = \pi/6,\ u = \sqrt{3}/2$

$$\int_{\pi/6}^{\pi/4} \frac{\sin x}{\cos^2 x}\ dx \rightarrow \int_{\sqrt{3}/2}^{1/\sqrt{2}} \frac{-1}{u^2}\ du = \left[\frac{1}{u}\right]_{\sqrt{3}/2}^{1/\sqrt{2}} = \sqrt{2} - \frac{2}{\sqrt{3}}$$

In MIA textbook - Exercise 7.4, all of question 1, Q2 is extension

In Leckie and Leckie - Exercise 3I page 93 Q1 to 7.

Lesson 4 - Integration by substitution with more complex examples

For an integral in the form	Use the substitution
$\int \sqrt{a - x^2}\, \mathrm{dx}$	$x = \sqrt{a} \sin u$, $\quad u = \sin^{-1} \frac{x}{\sqrt{a}}$
$\int \frac{1}{a^2 + x^2}\, dx$	$x = a \tan u$, $\quad u = \tan^{-1} \frac{x}{a}$
$\int kx\sqrt{a^2 + x^2}\, dx$	$u^2 = a^2 + x^2$, $\quad u = \sqrt{a^2 + x^2}$

These trigonometric identities are also supposed to help!

$\sin^2 x + \cos^2 x = 1 \qquad \sin 2x = 2 \sin x \cos x \qquad$ **$\tan^2 x + 1 = \sec^2 x$**

$\cos 2x = \cos^2 x - \sin^2 x \qquad \cos 2x = 2\cos^2 x - 1 \qquad \cos 2x = 1 - 2\sin^2 x$

Example 1

$\int \frac{3}{16 + x^2} dx$	Form 2, so $x = 4 \tan u$, $dx = 4 \sec^2 u \; du$
$\int \frac{3}{16 + (4 \tan u)^2} \times 4 \sec^2 u \; du$	
$\int \frac{3}{16 + 16 \tan^2 u} \times 4 \sec^2 u \; du$	
$\int \frac{12 \sec^2 u}{16(1 + \tan^2 u)} \; du$	$\tan^2 x + 1 = \sec^2 x$
$\frac{12}{16} \int \frac{\sec^2 u}{\sec^2 u} \; 4du$	
$\frac{3}{4} \int 1 \; du$	
$\frac{3}{4} u + C$	$x = 4 \tan u$, $\quad so \quad u = \tan^{-1} \frac{x}{4}$
$\frac{3}{4} \tan^{-1} \left(\frac{x}{4}\right) + C$	

Example 2

$\int x\sqrt{2-x^2}\ dx$	$u^2 = 2 - x^2$, $2u\,du = -2x\,dx$, $-u\,du = x\,dx$
$\int -u\sqrt{u^2}\ du$	
$\int -u^2\ du$	
$-\frac{1}{3}u^3 + c$	$u^2 = 2 - x^2$, $u = \sqrt{2-x^2}$
$-\frac{1}{3}(2-x^2)^{\frac{3}{2}} + c$	

For some examples, you will need substitution for, trig identities and a second substitution

Example 3 ***two substitutions***

$\int \frac{1}{x^2\sqrt{1+x^2}}$	$x = \tan u$, $dx = \sec^2 u$
$\int \frac{\sec^2 u\ du}{\tan^2 x\sqrt{1+\tan^2 x}}$	
$\int \frac{\sec^2 u}{\tan^2 x\sqrt{\sec^2 u}}\,du$	$\tan^2 x + 1 = \sec^2 x$
$\int \frac{\sec^2 u}{\tan^2 x\sec^2 u}\,du$	
$\int \frac{\sec u}{\tan^2 x}\,du$	
$\int \frac{1}{\cos u} \times \frac{\cos^2 u}{\sin^2 u}\,du$	
$\int \frac{\cos u}{\sin^2 u}\,du$	$v = \sin u$, $dv = \cos u\,du$
$\int \frac{1}{v^2}\,dv$	
$-\frac{1}{v} + c = -\frac{1}{\sin u} + C$	$u = \tan^{-1} x$
$-\frac{1}{\sin(\tan^{-1} x)} + C$	

In MIA textbook - Exercise 7.3, one or two questions from page 7 will do

In Leckie and Leckie - Exercise B for trig identities, Exercise 3G Q2 and Ex 3J

Lesson 5 – Inverse trig functions

By reversing the derivatives for inverse trig functions, we get:

$f(x)$	$F(x)$
$\int \frac{dx}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{dx}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Example 1

$$\int \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + C\,, \qquad \text{and} \qquad \int \frac{dx}{\sqrt{3 - 2x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

Example 2

$$\int \frac{dx}{16 + x^2} = \frac{1}{4}tan^{-1}\left(\frac{x}{4}\right) + C\,, \qquad \text{and} \qquad \int \frac{dx}{5 + x^2} = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

Take care each time to compare your integral with the standard

Example 3

$$\int \frac{dx}{\sqrt{8 - 2x^2}} = \int \frac{dx}{\sqrt{2}\sqrt{4 - x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{4 - x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x}{2}\right) + C$$

Calculating the definite integral:

Example 6

$$\int_0^4 \frac{dx}{9 + x^2} = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^4 = \frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) - \frac{1}{3}\tan^{-1} 0 = 0.309$$

In the MIA Textbook - Exercise 7.6:

Q1 and Q2 are necessary; Q3 and Q4 calculate definite integrals so are useful; Q5 uses form 3 and is suitable for extension; Q6 & Q7 use completed square form.

In Leckie and Leckie – Exercise 3E on page 84

Lesson 6 – Common forms

Some substitutions are so common that if you can identify their form, then you can just write the integrals. Remember that $F(x)$ is the anti-derivative of $f(x)$

form 1 $$\int f(ax+b)\ dx = \frac{1}{a}F(ax+b)\ + C$$

form 2 $$\int f'(x)f(x)\ dx = \frac{1}{2}(fx)^2\ + C$$

form 3 $$\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)|\ + C$$

Form 1 examples

$$\int \frac{1}{3x-1}\ dx = \frac{1}{3}\ln(3x-1)\ + C$$

$$\int \sec^2(2-x)\ \ dx = -\tan(2-x) + C$$

$$\int \sqrt{4x+1}\ \ dx = \frac{1}{4}(4x+1)^{\frac{3}{2}} \times \frac{2}{3} = \ \frac{1}{6}(4x+1)^{\frac{3}{2}} + C$$

Form 2 and Form 3 examples – be careful to identify $f(x)$

$$\int (2x+1)(x^2+x-6)\ dx \ \ = \frac{1}{2}(x^2+x-6)^2 + C$$

$$\int \frac{2\ln x}{x}\ \ dx = 2\int \frac{1}{x} \times \ln x\ \ dx = \frac{2}{2}(\ln x)^2 = \ln^2 x + C$$

$$\int \frac{\sec^2 x}{\tan x}\ \ dx = \ln|\tan x|\ + C$$

$$\int \frac{3(x+1)^2}{(x+1)^3}\ \ dx = \ln|(x+1)^3|\ + C$$

$$\int \frac{6x}{x^2+5}\ \ dx = 3\int \frac{2x}{x^2+5}\ \ dx = 3\ln|x^2+5|\ + C$$

In MIA textbook - Exercise 7.5: Form 1 – Question 1, Form 2 – Question 3, Form 3 – Question 5
In Leckie and Leckie - Exercise 3H Q1,2,4 and 7

Lesson 7 – Partial Fractions

For a partial with:

1. **a linear denominator - use common form 1**

$$\int \frac{2}{x}\ dx = 2\int \frac{1}{x}\ dx = 2\ln|x| + C \quad \text{or} \quad \int \frac{1}{3x-1}\ dx = \frac{1}{3}\ln|3x-1| + C$$

2. **a repeated denominator - use**

$$\int \frac{4}{(x+1)^2}\ dx = 4\int (x+1)^{-2}\ dx = -\frac{4}{x+1} + C$$

3. **an irreducible quadratic denominator - use form 3 and/or $\tan^{-1} x$**

$$\int \frac{4x+1}{x^2+2}\ dx = \int \frac{4x}{x^2+2}\ dx + \int \frac{1}{x^2+2} dx$$

$$= 2\int \frac{2x}{x^2+2}\ dx + \int \frac{1}{x^2+2} dx$$

$$= 2\ln|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Also note that an answer in the form $3\ln|x+1| - \ln|x-2| + C$

Can be rearranged using the laws of logs to $\ln\left(\frac{(x+1)^3}{x-2}\right) + C$

Example 1 Integrate

$$\int \frac{x+11}{(x+4)(3-x)}\ dx$$

Partial Fractions

$$\frac{x+11}{(x+4)(3-x)} = \frac{A}{x+4} + \frac{B}{3-x} \rightarrow \frac{x+11}{(x+4)(3-x)} = \frac{A(3-x)+B(x+4)}{(x+4)(3-x)}$$

$x = -4, \quad 7 = 7A, \quad \boldsymbol{A = 1} \qquad x = 3, \quad 14 = 7B, \quad \boldsymbol{B = 2}$

$$\frac{x+11}{(x+4)(3-x)} = \frac{1}{x+4} + \frac{2}{3-x} = \frac{1}{x+4} - \frac{2}{x-3}$$

Integration

$$\int \frac{x+11}{(x+4)(3-x)}\,dx = \int \frac{1}{x+4} - \frac{2}{x-3}\,dx$$

$$= \ln|x+4| - 2\ln|x-3| + C \quad \textbf{or} \quad \ln\left(\frac{(x+4)}{(x-3)^2}\right) + C$$

Example 2 Past Paper 2007 Q4

Given that $\int_4^6 \frac{2x^2-9x-6}{x(x-3)(x+2)}\,dx = \ln\left(\frac{m}{n}\right)$. Determine values for m and n **6**

Partial Fractions

$$\frac{2x^2-9x-6}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$2x^2-9x-6 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

If $x=0$, $\quad -6 = -6A$, $\quad \boldsymbol{A = 1}$, If $x=3$, $\quad -15 = 15B$, $\quad \boldsymbol{B = -1}$

If $x=-2$, $\quad 20 = 10C$, $\quad \boldsymbol{C = 2}$ $\qquad \frac{1}{x} - \frac{1}{x-3} + \frac{2}{x+2}$

$$\frac{2x^2-9x-6}{x(x-3)(x+2)} = \frac{1}{x} - \frac{1}{x-3} + \frac{2}{x+2}$$

Definite Integral

$$\int_4^6 \frac{2x^2-9x-6}{x(x-3)(x+2)}\,dx = \int_4^6 \frac{1}{x}\,dx - \int_4^6 \frac{1}{x-3}\,dx + 2\int_4^6 \frac{1}{x+2}\,dx$$

$$= \left[\ln(x) - \ln(x-3) + 2\ln(x+2)\right]_4^6$$

$$= (\ln 6 - \ln 3 + 2\ln 8) - (\ln 4 - \ln 1 + 2\ln 6)$$

$$= \ln\left(\frac{6\times 8^2\times 1}{3\times 4\times 6^2}\right) = \ln\left(\frac{8}{9}\right)$$

$$\mathbf{m = 8}, \qquad \mathbf{n = 9}$$

In MIA textbook - Exercise 7.7 Q1 – 6 is sufficient practice to be able to tackle past papers.
In Leckie and Leckie - Exercise 3F do the whole exercise.

Lesson 8 - Integration by parts

Integration by parts is used to integrate the product of two functions

Give two functions in the form $f(x) \times g(x)$ then the rule for integration is

$$\int f(x) \times g(x) \;\; dx = f(x) \times \int g(x)\, dx - \int \left[\int g(x)\, dx\right] \times f'(x) dx$$

Or using the shorthand $\boldsymbol{f \cdot g}$

$$\int \boldsymbol{f} \cdot \boldsymbol{g} \;\; dx = \boldsymbol{f} \cdot \int \boldsymbol{g}\, dx - \int \left[\int \boldsymbol{g}\, dx\right] \cdot \boldsymbol{f}' dx$$

Be careful when choosing $f(x)$ and $g(x)$:

- Always assign $f(x)$ to a function which simplifies as it differentiates
- Never assign $g(x)$ to a function with no anti-derivative such as $\ln x$

Example 1

$$\int \boldsymbol{x} \cdot \cos \boldsymbol{x} \;\; dx = \boldsymbol{x} \cdot \int \cos \boldsymbol{x}\, dx - \int \left[\int \cos \boldsymbol{x}\, dx\right] \cdot \mathbf{1} dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= \boldsymbol{x \sin x + \cos x + C}$$

Example 2

$$\int \ln x \cdot x^2 \;\; dx = \ln x \cdot \int x^2\, dx - \int \left[\int x^2\, dx\right] \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3} x^2\, dx$$

$$= \boldsymbol{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

In this case $f(x) = \ln x$ as we cannot integrate $\ln x$

Example 3

$$\int (1-2x)e^x \; dx = (1-2x)\int e^x\,dx - \int \left[\int e^x\,dx\right] -2 \; dx$$

$$= (1-2x)e^x - \int -2e^x\,dx$$

$$= (1-2x)e^x + 2\int e^x\,dx$$

$$= (1-2x)e^x + 2e^x + C$$

$$= 3e^x - 2xe^x + C \quad \text{or} \quad (3-2x)e^x + C \quad \text{etc}$$

Example 4

$$\int 2x\sec^2 x \; dx = 2x\int \sec^2 x\,dx - \int \left[\int \sec^2 x\,dx\right] 2 \; dx$$

$$= 2x\tan x - \int 2\tan x\,dx$$

$$= 2x\tan x - \int 2\frac{\sin x}{\cos x}dx$$

$$= 2x\tan x + 2\int \frac{-\sin x}{\cos x}dx$$

As $\int \frac{-\sin x}{\cos x}dx$ takes the form $\int \frac{f'(x)}{f(x)}dx$, we can use common form 3

$$= \mathbf{2x\tan x + 2\ln|\cos x| + C}$$

Example 5

$$\int \sin x\ln(\cos x) \; dx = \ln(\cos(x))\int \sin x\,dx - \int \left[\int \sin x\,dx\right]\frac{-\sin x}{\cos x} \; dx$$

$$= -\cos x\ln(\cos x)) - \int -\cos\; x \cdot \frac{-\sin x}{\cos x}dx$$

$$= -\cos x\ln(\cos x)) - \int \sin\; x\,dx$$

$$= \cos x \; - \cos x\ln(\cos x)) + C$$

***In MIA textbook* - Exercise 7.8 - Q1 to 3 is sufficient, Q 4 is not necessary**

In Leckie and Leckie - Exercise 3K on page 98, do the whole exercise!

Lesson 9 - Integration by parts continued

To find the integral of a function where only the derivative is known – such as $\int \ln(x)\ dx$. You multiply in an extra function $f(x) = 1$ and apply integration by parts

Example 1

$$\int \ln \boldsymbol{x}\ \ dx = \int \mathbf{1} \cdot \ln \boldsymbol{x}\ \ dx = \ln x \int 1\, dx - \int \left[\int 1\, dx\right] \times \frac{1}{x} dx$$

$$= x \ln x - \int 1\, dx$$

$$= \boldsymbol{x \ln x - x + C}$$

This is a simple example, other examples such as $\ln(2x + 1)$ get very difficult very quickly

<u>When integration by parts must be applied more than once:</u>

Example 1

$$\int x^2 \cos x\ \ dx = x^2 \int \cos x\, dx - \int \left[\int \cos x\, dx\right] 2x\ \ dx$$

$$= x^2 \sin x\ - \int 2x \sin x\, dx$$

Working only on $\int 2x \sin x\, dx$

$$\int 2x \sin x\, dx = -2x \cos x - \int -2\cos\ x\, dx$$

$$= -2x \cos x + 2 \sin x$$

So

$$\int x^2 \cos x\ \ dx = x^2 \sin x - \ (-2x \cos x + 2 \sin x)$$

$$= \boldsymbol{x^2 \sin x + 2x \cos x - 2 \sin x + C}$$

Remember to subtract your second application of integration by parts

Integration by parts where the original integral on the RHS and you appear to be stuck in a loop:

Example 2

$$\int \sin x\, e^x \; dx = \sin x \int e^x\, dx - \int \left[\int e^x\, dx\right] \times \cos x\, dx$$

$$\int \sin x\, e^x \; dx = \sin x \; e^x - \int \cos x \; e^x\, dx$$

$$\int \sin x\, e^x \; dx = \sin x\, e^x - \left(\cos x \int e^x\, dx - \int e^x \times -\sin x\, dx\right)$$

$$\int \sin x\, e^x \; dx = \sin x\, e^x - \left(\cos x\, e^x + \int \sin x\, e^x\, dx\right)$$

$$\int \sin x\, e^x \; dx = \sin x\, e^x - \cos x\, e^x - \int \sin x\, e^x\, dx$$

Rearrange

$$\int \sin x\, e^x\, dx + \int \sin x\, e^x\, dx = e^x(\sin x \; - \cos x)$$

$$2\int \sin x\, e^x\, dx = e^x(\sin x \; - \cos x)$$

$$\int \sin x\, e^x\, dx = \frac{e^x}{2}(\sin x \; - \cos x) + C$$

Remember that you can use https://www.integral-calculator.com/ to check your answers

In MIA textbook - Exercise 7.8 Q5 and Exercise 7.9 Q2 will be sufficient.
In Leckie and Leckie - Exercise 3L on page 101, do the whole exercise!