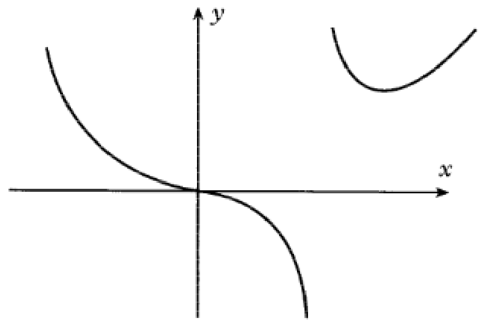
Calculus - Question by Topic		
	Differentiation	
1	For $f(x) = x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer	**3**
2	Given that $f(x) = \sqrt{x}e^{-x}$, obtain and simplify $f'(x)$	**4**
3	Differentiate $g(x) = \frac{\tan^{-1} 2x}{1+4x^2}$	**3**
4	Differentiate $y = \frac{1+\ln x}{3x}$	**3**
5	Given $y = (x+1)^2(x+2)^4$ and $x > 0$, use logarithmic differentiation to show that $\frac{dy}{dx}$ can be expressed in the form $\left(\frac{a}{x+1} + \frac{b}{(x+2)}\right)y$, stating the values of constants a and b	**3**
6	A curve is defined by the equation $y^3 + 3xy = 3x^2 - 5$. The curve passes through the point $A(2,1)$, find the equation of the tangent to the curve at this point	**4**
7	A curve is defined by the equation $2y^2 - 2xy - 4y + x^2 = 0$. Obtain the x-coordinate of each point at which the curve has a horizontal tangent	**4**
8	Given that $x = \sqrt{t}$ and $y = t^3 - \frac{5}{2}t^2$ for $t > 0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of *t* in simplified form Show that $\frac{d^2y}{dx^2} = at^2 + bt$, determining the values of the constants *a* and *b*	**4** **3**
9	A curve is defined by the equations $x = 5\cos\theta \quad y = 5\sin\theta\,, \quad 0 < t < 2\pi$ (a) Use parametric differentiation to find $\frac{dy}{dx}$ in terms of θ (b) Hence find the equation of the tangent to the curve when $\theta = \frac{\pi}{4}$	**2** **3**

	Differential Equations – first order and homogeneous second order $(= \mathbf{0})$	
10	The volume $V(t)$ of a cell at time t changes according to the law $\frac{dV}{dt} = V(10 - V) \quad$ for $0 < V < 10$ (a) Show that $\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$ for some constant C (b) Also given that $V(0) = 5$, show that $V(t) = \frac{10e^{10t}}{1+e^{10t}}$ (c) Obtain the limiting value of $V(t)$ $as\ t \to \infty$	**4** **3** **2**
11	Functions $x(t)$ and $y(t)$ satisfy $\frac{dy}{dt} = -xy^2$, $\frac{dx}{dt} = -x^2y$ When $t = 0$, $x = 1$ and $y = 2$ Express $\frac{dy}{dx}$ in terms of x and y and hence obtain y as a function of x	**5**
12	A mathematical biologist believes that the differential equation $x\frac{dy}{dx} - 3y = x^4$ models a process. Find the general solution of this differential equation Given that $y = 2$ when $x = 1$, find the particular solution, expressing y in terms of x The biologist subsequently decides that a better model is given by the equation $y\frac{dy}{dx} - 3x = x^4$. Given that $y = 2$ when $x = 1$, obtain y in terms of x	**5** **2** **4**
13	Find the solution $y = f(x)$ to the differential equation $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$ Given that $y = 4$ and $\frac{dy}{dx} = 3$ when $x = 0$	**6**
14	Obtain the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$ Hence find the particular solution for which: $y = 3$ when $x = 0$ and $y = e^{-\pi}$ when $x = \frac{\pi}{2}$	**4** **3**

	Integration	
15	Find $$\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\ dx$$	**3**
16	(a) Show that $$\int \sin^2 x \cos^2 x \quad dx = \int \cos^2 x\ dx - \int \cos^4 x\ dx$$	**2**
	(b) Show that $$\int_0^{\frac{\pi}{4}} \cos^2 x\ dx\ = \frac{\pi + 2}{8}$$	**3**
17	Use the substitution $u = 1 + x$, to obtain the exact value of $$\int_0^3 \frac{x}{\sqrt{1+x}} dx$$	**5**
18	Use the substitution $x = 1 + \sin\theta$, to evaluate $$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1 + \sin\theta)^3}\ d\theta$$	**5**
19	Evaluate $$\int_0^1 \frac{1}{x^2 - x - 6}\ dx$$	**6**
20	Use integration by parts of find $$\int x^2 \sin x\ dx$$	**4**
21	Use integration by parts to obtain the exact value for $$\int_0^1 xe^{-x}\ dx$$	**3**

Properties of functions

22 The diagram shows part of the graph of $y = \frac{x^3}{x-2}$, $x \neq -2$



(a) Write down the equation of the vertical asymptote **1**

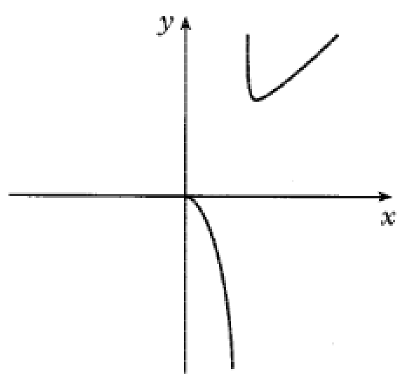
(b) Find the coordinates of the stationary points of the graph of $y = \frac{x^3}{x-2}$ **4**

(c) Write down the coordinates of the stationary points of the graph of $y = \left|\frac{x^3}{x-2}\right| + 1$ **2**

23 The diagram shows part of the graph of a function which satisfies the following conditions

(a) f is an even function

(b) Two of the asymptotes of the graph $y = f(x)$ are $y = x$ and $x = 1$



Copy the diagram and complete the graph.

Write down equations for the other two asymptotes **3**

24 Determine whether $f(x) = x^2 \sin x$ is odd, even or neither. **3**

Fully justify your answer

25		
	(a) Express $\frac{x^2}{(x+1)^2}$ in the form $$A + \frac{B}{x+1} + \frac{C}{(x+1)^2}, \qquad x \neq -1$$ stating the values of the constants A, B and C.	**3**
	(b) A curve is defined by $y = \frac{x^2}{(x+1)^2}$	
	(i) write down the equations of the asymptotes	**2**
	(ii) Find the stationary point and justify its nature	**2**
	(iii) Identify any points of inflexion	**2**
	(iv) Sketch the curve clearly marking the features found above	**3**

Calculus - Answers		
	Differentiation	
1	$f'(x) = (1+x)^{10} + 10x(1+x)^9$ $= (1+x)^9(1+x+10x) = \mathbf{(1+11x)(1+x)^9}$	**3**
2	$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^{-x} + \sqrt{x} \times -e^{-x}$ $= \frac{e^{-x}}{2\sqrt{x}} - \sqrt{x}e^{-x}$ $= \frac{(e^{-x} - 2xe^{-x})}{2\sqrt{x}}$ $= \boldsymbol{\frac{e^{-x}(1-2x)}{2\sqrt{x}}}$	**4**
3	$g'(x) = \frac{\frac{2}{1+4x^2} \times 1 + 4x^2 - \tan^{-1} 2x\ \ (8x)}{(1+4x^2)^2}$ $\boldsymbol{g'(x) = \frac{2 - 8x\, tan^{-1} 2x}{(1+4x^2)^2}}$	**3**
4	$\frac{dy}{dx} = \frac{\left(\frac{1}{x}\right) 3x - 3(1+\ln x)}{9x^2} = \frac{-3\ln x}{9x^2} = -\boldsymbol{\frac{\ln x}{3x^2}}$	**3**
5	$y = (x+1)^2(x+2)^{-4}$ $\ln y = \ln((x+1)^2(x+2)^{-4})$ $\ln y = 2\ln(x+1) - 4\ln(x+2)$ $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+1} - \frac{4}{x+2}$ $\frac{dy}{dx} = \left(\frac{2}{x+1} - \frac{4}{x+2}\right)y$ $\boldsymbol{a = 2}, \quad \boldsymbol{b = -4}$	**3**
6	$3y^2\frac{dy}{dx} + 3y + (3x)\frac{dy}{dx} = 6x$ $(3y^2 + 3x)\frac{dy}{dx} = 6x - 3y$ $\frac{2x - y}{y^2 + x} = \frac{dy}{dx}$ At A (2,1) the gradient is $\frac{4-1}{1+2} = \frac{3}{3} = 1$ equation of the tangent $\boldsymbol{y = x - 1}$	**4**

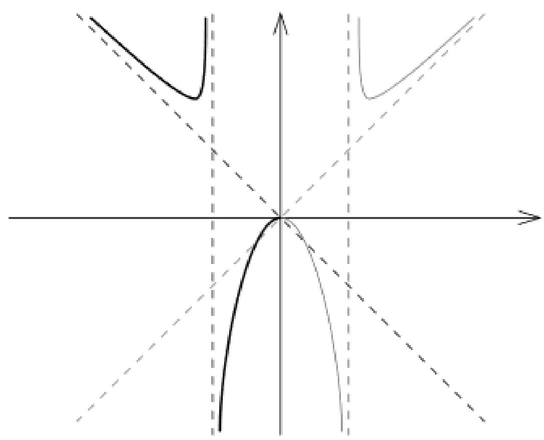
7	$\frac{d}{dx}(2y^2) - \frac{d}{dx}(2xy) - \frac{d}{dx}(4y) + \frac{d}{dx}(x^2) = 0$ $4y\frac{dy}{dx} - \left(2y + (2x)\frac{dy}{dx}\right) - 4\frac{dy}{dx} + 2x = 0$ $(4y - 2x - 4)\frac{dy}{dx} = 2y - 2x$ $\frac{dy}{dx} = \frac{2y - 2x}{4y - 2x - 4}$ For a horizontal tangent $2y - 2x = 0,\ so\ y = x$ Substituting into the original curve this gives $2y^2 - 2y^2 - 4y + y^2 = 0 \rightarrow\ y^2 - 4y = 0,\ \ \boldsymbol{y = 0}$ **and** $\boldsymbol{y = 4}$	**4**
8	$\frac{dy}{dt} = 3t^2 - 5t \quad \frac{dx}{dt} = \frac{1}{2\sqrt{t}},$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (3t^2 - 5t) \times 2\sqrt{t} = \boldsymbol{6t^{\frac{5}{2}} - 10t^{\frac{3}{2}}},$ $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(6t^{\frac{5}{2}} - 10t^{\frac{3}{2}}\right) \times \frac{dt}{dx} = \left(15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}\right) \times 2\sqrt{t} = 30t^2 - 30t$ $\boldsymbol{a = 30}$ and $\boldsymbol{b = -30}$	**4** **3**
9	$\frac{dy}{dt} = 5\cos\theta \quad \frac{dx}{dt} = -5\sin\theta, \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{5\cos\theta}{-5\sin\theta} = -cot\theta,$ When $t = \frac{\pi}{4},\ x = \frac{-5}{\sqrt{2}},\ y = \frac{5}{\sqrt{2}},\ m = -1$ tangent is $\boldsymbol{y = -x + 5\sqrt{2}}$	**2** **3**

	Differential Equations – first order and homogeneous second order $(=\mathbf{0})$	
10	$$\frac{1}{V(10-V)}\frac{dV}{dt}=1,$$ Using partial fractions $$\frac{1}{V(10-V)}=\frac{A(10-V)+BV}{V(10-V)}=\frac{1}{10V}+\frac{1}{(10-v)}$$ $$\int\left(\frac{1}{10V}+\frac{1}{10(10-v)}\right)dV=\int 1\ dt$$ $$\boldsymbol{\frac{1}{10}\ln V-\frac{1}{10}\ln(10-V)=t+C}$$ **as required** $$V(0)=5\qquad \frac{1}{10}\ln 5-\ln(5)=0+C,\quad C=0$$ $$\ln V-\ln(10-V)=10t$$ $$\ln\left(\frac{V}{10-V}\right)=10t$$ $$\frac{V}{10-V}=e^{10t}$$ $$V=10e^{10t}-Ve^{10t}$$ $$V(1+e^{10t})=10e^{10t}$$ $$\boldsymbol{V=\frac{10e^{10t}}{1+e^{10t}}}$$ **as required** $$V=\frac{10}{e^{-10t}+1}\quad \textbf{this}\rightarrow \boldsymbol{10\ as\ t}\rightarrow\infty$$	**4** **3** **2**
11	$$\frac{dy}{dx}=\frac{-xy^2}{-x^2y},\qquad \frac{dy}{dx}=\frac{y}{x},\quad \text{so}\ \int\frac{1}{y}\,dy=\int\frac{1}{x}\,dx$$ $$\ln y=\ln x+c,$$ Given $x=1$ and $y=2$, $\ln 2=\ln 1+C,\ C=\ln 2$ $$\ln y=\ln x+\ln 2,\quad \ln y=\ln 2x,\quad \text{thus}\ \boldsymbol{y=2x}$$	**5**

12	$$\frac{dy}{dx}-\frac{3}{x}y=x^3$$ $$P=-\frac{3}{x},\qquad \int P=\ -3\ln x=\ln x^{-3},\ \ I=e^{\ln x^{-3}}=x^{-3}$$ $$Iy=\int I\times x^4\,dx$$ $$Iy=\int x\,dx$$ $$\frac{y}{x^3}=\frac{1}{2}x^2+C,\ \ \boldsymbol{y=\frac{1}{2}x^5+Cx^3}$$ $$2=1+C,\quad C=1,\ \text{so}\ \ \boldsymbol{y=x^4+x^3}$$ $$y\frac{dy}{dx}=x^4+x^3$$ $$\int y\,dy=\int x^4+3x\,dx$$ $$\frac{1}{2}y^2=\frac{1}{5}x^5+\frac{3}{2}x^2+C$$ $$\frac{1}{2}\times 2^2=\frac{1}{5}+\frac{3}{2}+C,\qquad C=\frac{3}{10}$$ $$\boldsymbol{y=\sqrt{\frac{2}{5}x^5+3x^2+\frac{3}{5}}}$$	**5** **2** **4**
13	The auxiliary equation is $4m^2-4m+1=0,\quad m=\frac{4}{8}=\frac{1}{2}$ The general solution is $y=Ae^{0.5x}+Bxe^{0.5x}$ $y=4$ when $x=0$ $\qquad 4=A+0,\quad A=4$ $\frac{dy}{dx}=3$ when $x=0$ $\qquad \frac{dy}{dx}=\frac{1}{2}Ae^{0.5x}+Be^{0.5x}+\frac{1}{2}Bxe^{0.5x}$ $3=\frac{1}{2}A+B,\ \ B=1$ The particular solution is $\boldsymbol{y=4e^{0.5x}+xe^{0.5x}}$	**6**
14	The auxiliary equation is $m^2+4m+5=0,\ \ (m+2)^2+1,\quad m=\ -2\pm i$ The general solution is $y=e^{-2x}\,(A\cos x+B\sin x)$ $y=3$ when $x=0$ $\qquad 3=1(a+0),\ \ A=3$ $y=e^{-\pi}$ when $x=\frac{\pi}{2}$ $\qquad e^{-\pi}=e^{-\pi}(0+B),\quad B=1$ The particular solution is $\boldsymbol{y=e^{-2x}\,(3\cos x+\sin x)}$	**4** **3**

	Integration	
15	$$f(x)=x^4-x^2+1,\quad f'(x)=4x^3-2x,$$ $$\int \frac{12x^3-6x}{x^4-x^2+1}\,dx = 3\int \frac{4x^3-2x}{x^4-x^2+1}\,dx$$ $$= \mathbf{3\ln(x^4-x^2+1)+C}$$	**3**
16 (a)	$$\sin^2 x+\cos^2 x=1,\qquad \sin^2 x = 1-\cos^2 x$$ $$\int \sin^2 x\cos^2 x\ \ dx = \int (1-\cos^2 x)\cos^2 x\ dx$$ $$\int \sin^2 x\cos^2 x\ \ dx = \int \mathbf{\cos^2 x\ dx} - \int \mathbf{\cos^4 x\ dx}$$	**2**
16 (b)	$$\cos 2x = 2\cos^2 x-1,\quad \cos^2 x=\frac{1}{2}(\cos 2x+1)$$ $$\int_0^{\frac{\pi}{4}} \cos^2 x\ dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 2x)\ \ dx$$ $$\frac{1}{2}\int_0^{\frac{\pi}{4}} 1+\cos 2x\ dx = \frac{1}{2}\left[x+\frac{1}{2}\sin 2x\ \right]_0^{\frac{\pi}{4}}$$ $$=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\sin\left(\frac{\pi}{2}\right)-0\right)$$ $$=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)=\mathbf{\frac{\pi+2}{8}}$$	**3**
17	New limits are, when $x=3$, $u=4$, when $x=0$, $u=1$ If $u=1+x\ \ \rightarrow du=dx$ **and** $x=u-1$ $$\int_0^3 \frac{x}{\sqrt{1+x}}dx = \int_1^4 \frac{u-1}{\sqrt{u}}\ du$$ $$\int_1^4 \frac{u-1}{\sqrt{u}}\ du = \int_1^4 u^{\frac{1}{2}}-u^{-\frac{1}{2}}\ du$$ $$=\left[\tfrac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^4=\left(\tfrac{4}{3}\right)-\left(-\tfrac{4}{3}\right)=\mathbf{\tfrac{8}{3}}$$	**5**

18	New limits for integration are; when $\theta = \frac{\pi}{2}, x = 2,$ when $\theta = 0,\ x = 1$ If $x = 1 + \sin\theta \quad \to dx = \cos\theta\, d\theta$ $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)^3}\, d\theta = \int_1^2 \frac{1}{x^3}\, dx$ $\int_1^2 \frac{1}{x^3}\, dx = \left[-\frac{1}{2x^2}\right]_1^2$ $= \left(-\frac{1}{8}\right) - \left(-\frac{1}{2}\right) = \mathbf{\frac{3}{8}}$	**5**
19	Partial Fractions $\frac{1}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)} = \frac{1}{5(x-3)} - \frac{1}{5(x+2)}$ Integration $\int_0^1 \frac{1}{(x-3)(x+2)}\, dx = \frac{1}{5}\int_0^1 \left(\frac{1}{x-3} - \frac{1}{x+2}\right) dx$ $= \frac{1}{5}[\ln(x-3) - \ln(x+2)\,]_0^1$ $= \frac{1}{5}(\ln(-2)\ - \ln 3 - \ln(-3) + \ln 2)$ $= \frac{1}{5}\ln\left(\frac{-2\times 2}{3\times -3}\right) = \frac{1}{5}\ln\left(\frac{4}{9}\right) = \mathbf{-0.162}$	**6**
20	$\int x^2 \sin x\, dx = -x^2\cos x - \int -2x\cos x\, dx$ $\int x^2 \sin x\, dx = -x^2\cos x + 2\left[x\sin x - \int \sin x\, dx\right]$ $\int x^2 \sin x\, dx = \mathbf{-x^2\cos x + 2x\sin x + 2\cos x + C}$	**4**
21	$\int_0^1 xe^{-x}\, dx = [-xe^{-x}\,]_0^1 + \int e^{-x} \times 1\, dx$ $= -e^{-1} - [e^{-x}\,]_0^1 = -e^{-1} - (e^{-1} - 1) = \mathbf{1 - \frac{2}{e} = 0.2642}$	**3**

	Properties of functions	
22	(a) There is a vertical asymptote at $\boldsymbol{x} = \mathbf{2}$ (b) $$f'(x) = \frac{3x^2(x-2) - x^3}{(x-2)^2} = \frac{2x^3 - 6x^2}{(x-2)^2} = \frac{2x^2(x-3)}{(x-2)^2}$$ When $\frac{2x^2(x-3)}{(x-2)^2} = 0$, $x = 0$ and $x = 3$, SP at $(\mathbf{0}, \mathbf{0},)$ **and** $(\mathbf{3}, \mathbf{27})$ (c) For $y = \left\lvert\frac{x^3}{x-2}\right\rvert + 1$, stationary points occur at $(\mathbf{0}, \mathbf{1})$ **and** $(\mathbf{3}, \mathbf{28})$	**1** **4** **2**
23	Other asymptotes are $y = -1$ and $x = -1$ 	**3**
24	An even function has the property that $\boldsymbol{f(-x) = f(x)}$ An odd function has the property that $\boldsymbol{f(-x) = -f(x)}$ $f(-x) = x^2\sin(-x) = -x^2\sin(x) = -f(x)$, so this is an **odd** function	**3**

25 By algebraic long division $\frac{x^2}{(x+1)^2} = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$ **3**

(i) $y = \frac{x^2}{(x+1)^2}$ has a vertical asymptote at $\boldsymbol{x = -1}$

$$y = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2} \quad tends \ \rightarrow 1 \ as \ x \rightarrow \infty$$

So there is a horizontal asymptote at $\boldsymbol{y = 1}$.

For ∞^- you will get $1 - (-0.00000a)$ so this approaches from above

For ∞^+ you will get $1 - (40.00000a)$ so this approaches from below **2**

(ii) $f'(x) = \frac{2x(x+1)^2 - 2x^2(x+1)}{(x+1)^4} = \frac{2x}{(x+1)^3} \neq 0$ **2**

$f'(x) = 0$ when $x = 0$

$$f''(x) = \frac{2(x+1)^3 - 6x(x+1)^2}{(x+1)^6} = \frac{2-4x}{(x+1)^4}, \qquad f''(0) = 2 > 0$$

$\boldsymbol{(0, 0)}$ is a minimum stationary point.

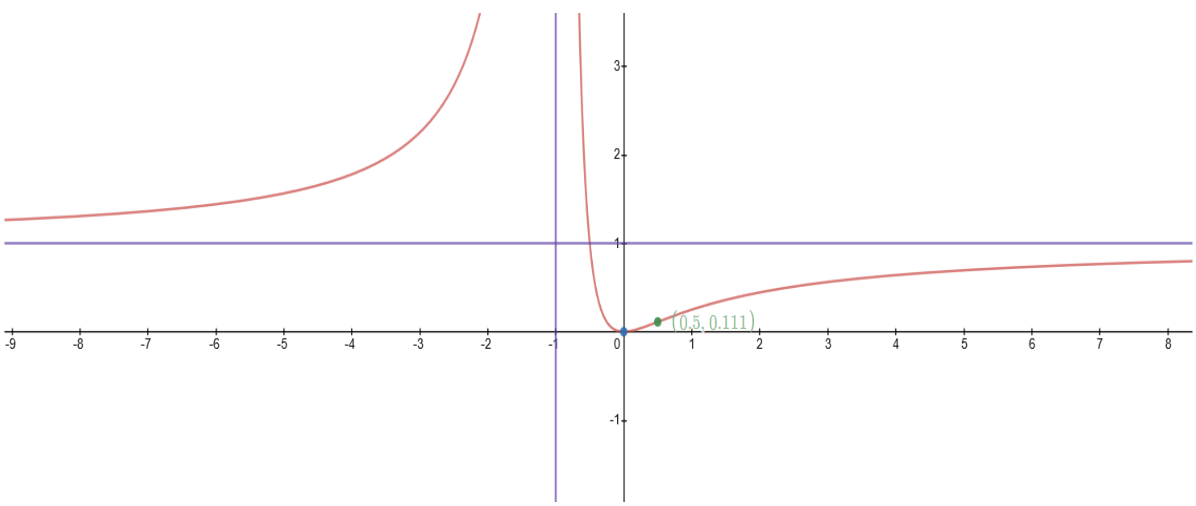
(iii) Check for points of inflexion $f''(x) = 0,\ 2 - 4x = 0,\ x = \frac{1}{2},$ **2**

x	$\rightarrow$	$\frac{1}{2}$	$\rightarrow$
$f''(x)$	positive	zero	Negative
Concavity	up		Down

possible POI at $\left(\frac{1}{2}, \frac{1}{9}\right)$

Change in concavity so this is a point of inflexion

(iv)



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