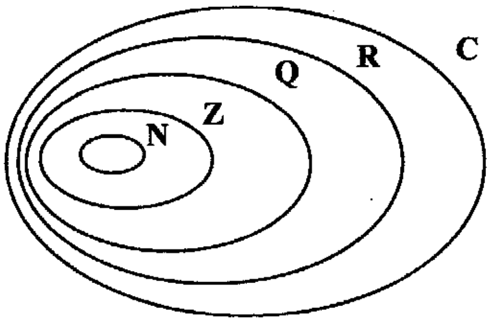
Complex Numbers

Working with natural numbers we can solve $2x = 8$.
Working with integers numbers we can solve $2x = -8$.
Working with rational numbers we can solve $2x = 7$.
Working with irrational numbers we can solve $x^2 = 7$.
We still cannot solve $x^2 = -1$ so we further extend the number system:

$$\boxed{\sqrt{-1} = i}$$

We now have a whole set of **imaginary** numbers *e.g.* $2i$, $5i$, $-3i$ and **complex numbers** *e.g.* $2 + 3i$ and $5 - 7i$ made up from a real and an imaginary part.

We now have a complete number system as and polynomial equation with complex coefficients has complex roots.



Complex numbers are written in the form $a + bi$ $a, b \in \Re$ *e.g.* $z = 5 - 7i$.
The real part of $z = \text{Re}(z) = 5$ and the imaginary part of $z = \text{Im}(z) = -7i$.

Learning to solve Quadratic Equations in the set of Complex Numbers

- Use the quadratic formula to find solutions to *all* quadratic equations.

Example
$z^2 + 6z + 10 = 0$

$$z = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 10}}{2 \times 1}$$

$$= \frac{-6 \pm \sqrt{-4}}{2}$$

$$= \frac{-6 \pm 2i}{2}$$

$z = -3 + i$ or $z = -3 - i$

Algebra: Complex Numbers

Learning to perform arithmetic operations on complex numbers

- Add, subtract and multiply complex numbers
- Find the complex conjugate
- Use the complex conjugate to divide complex numbers

Let $z_1 = 3+4i$ and $z_2 = 5-6i$

$z_1 + z_2 = 3+4i + 5-6i = 8-2i$

$z_1 - z_2 = 3+4i - (5-6i) = -2+10i$

$\to -1$

$z_1 z_2 = (3+4i)(5-6i) = 15-18i+20i-24i^2$

$= 39+2i$

The **complex conjugate** of z, $\bar{z}$ is a number such that the product $z\bar{z}$ is a real number.

Recall the difference of two squares $(x-3)(x+3) = x^2-9$

The complex conjugate of $(a+bi)$ is $(a-bi)$ because $(a+bi)(a-bi) = a^2 - b^2i^2$

$= a^2+b^2 \quad \in \Re$

$\bar{z}_2 = 5+6i$

$\frac{z_1}{z_2} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i}$

$= \frac{15+18i+20i+24i^2}{25-36i^2}$

$= \frac{-9+38i}{61}$

$= \frac{-9}{61} + \frac{38}{61}i$

We can only divide by a real number so start by multiplying top and bottom by $\bar{z}$.

p 207 – 209 Ex 12.1 and Ex 12.2

Algebra: Complex Numbers

Learning to solving Polynomial Equations

- Know that a polynomial of degree *n* will have *n* complex roots
- Know that roots occur in conjugate pairs
- Use polynomial division to factorise polynomials

Example

Solve $x^3 + x^2 + 3x - 5 = 0$

$$\begin{array}{c|cccc} 1 & 1 & 1 & 3 & -5 \\ & & 1 & 2 & 5 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array}$$

The remainder is zero so $x = 1$ is a root.

$$x^3 + x^2 + 3x - 5 = (x-1)(x^2 + 2x + 5)$$

$$x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 5}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$x = 1$ or $x = -1 + 2i$ or $x = -1 - 2i$

We can now find all three roots. One real $x = 1$ and two complex $x = -1 + 2i$ and $x = -1 - 2i$
The complex roots are complex conjugates.

The Fundamental Theorem of Algebra (Gauss, 1799)

- Every polynomial equation with complex coefficients has at least one root in the set of complex numbers.
- If a root is non-real then its complex conjugate is also a root.
- A polynomial of degree n will have n complex roots

Example

Show that $z = 2 + i$ is a root of $z^4 - 2z^3 - z^2 + 2z + 10 = 0$ and find the remaining roots.

$$
\begin{array}{r|rrrrr}
2+i & 1 & -2 & -1 & 2 & 10 \\
 & & 2+i & 2i+i^2 & -6+2i & -10 \\
\hline
 & 1 & i & -2+2i & -4+2i & \boxed{0}
\end{array}
$$

The remainder is zero so $z = 2+i$ is a root.

If $z = 2+i$ is a root then $z = 2-i$ is a root.

Factors: $(z-2-i)$ and $(z-2+i)$

$$(z-2-i)(z-2+i) = z^2 - 2z + zi - 2z + 4 - 2i - zi + 2i - i^2$$
$$= z^2 - 4z + 5$$

$$
\begin{array}{r l}
 & \quad\; z^2 + 2z + 2 \\
z^2 - 4z + 5 & \big) \; z^4 - 2z^3 - z^2 + 2z + 10 \\
 & \;\; \underline{z^4 - 4z^3 + 5z^2} \\
 & \qquad 2z^3 - 6z^2 + 2z + 10 \\
 & \qquad \underline{2z^3 - 8z^2 + 10z} \\
 & \qquad\qquad 2z^2 - 8z + 10 \\
 & \qquad\qquad \underline{2z^2 - 8z + 10} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

$$z^2 + 2z + 2 = 0$$

$$z = \frac{-2 \pm \sqrt{4 - 4\times 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

The remaining roots are $z = 2-i$, $z = -1+i$ and $z = -1-i$

p224 Ex 12.8 Q1, 2, 4, 6

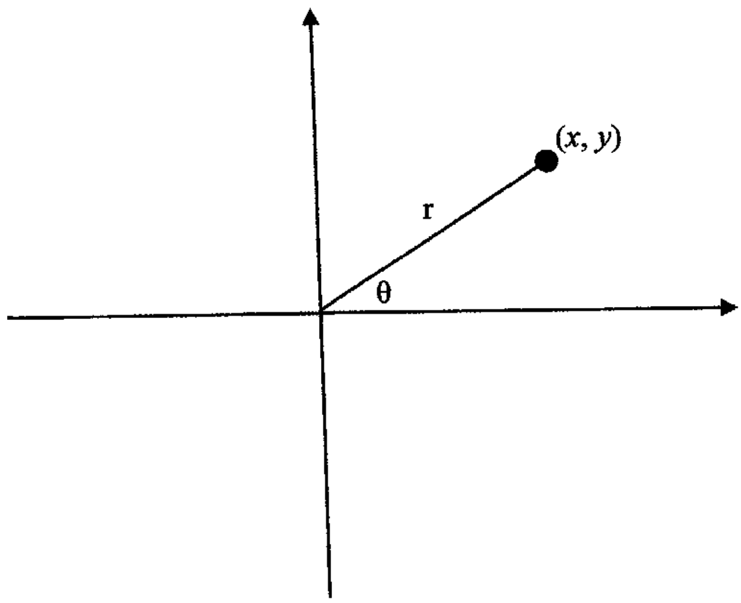
Algebra: Complex Numbers

Learning the Geometrical Interpretation of Complex Numbers

- Find the modulus and the argument of a complex number
- Convert between Cartesian and polar form

The complex number $z = x + yi$ can be represented on the **complex plane** by the point (x, y). In this context the x- axis is known as the real-axis and any point on this axis is a purely real number. The y-axis is the imaginary axis.
The resulting picture is known as an Argand diagram after the Swiss mathematician Jean Robert Argand.

Polar Coordinates
(x, y) are **cartesian coordinates.** Alternatively, the point P can reached by rotating anti-clockwise from the x-axis and moving out along a radius from O giving the polar coordinates (r, θ). Using polar coordinates for complex numbers allows some very powerful results.



$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\dfrac{y}{x}$

$x = r\cos\theta$

$y = r\sin\theta$

Applying polar coordinates to complex numbers
r is the **modulus** of z and is denoted by $|z|$.
θ is the **argument** of z and is denoted by arg z.

Multiple equivalent values for the argument are possible but the **principal argument** is in the range $-\pi < \theta \leq \pi$.

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

Example 1

Find the modulus and argument of $z = \sqrt{3} + i$ and show z on an argand diagram. Write z in polar form.

$|z| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$

$\arg z = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Example 2

Write $z = 2\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$ in Cartesian form using exact values.

$z = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= \sqrt{2} - \sqrt{2}i$

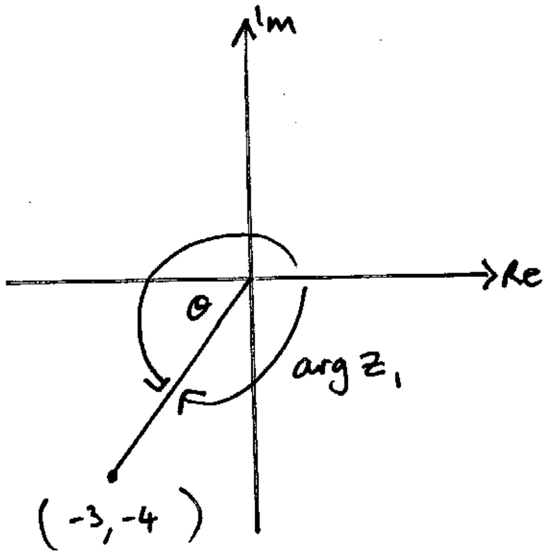
p 211 Ex 12.3 Q1-7

Algebra: Complex Numbers

Learning the properties of the modulus and argument of a complex number

- Know how to multiply complex numbers in polar form
- Know how to divide complex numbers in polar form
- Know De Moivre's Theorem

Plot $Z_1 = -3 - 4i$ on an argand diagram and find $|Z_1|$ and two different expressions for arg Z_1.



$|z_1| = \sqrt{3^2+4^2} = 5$

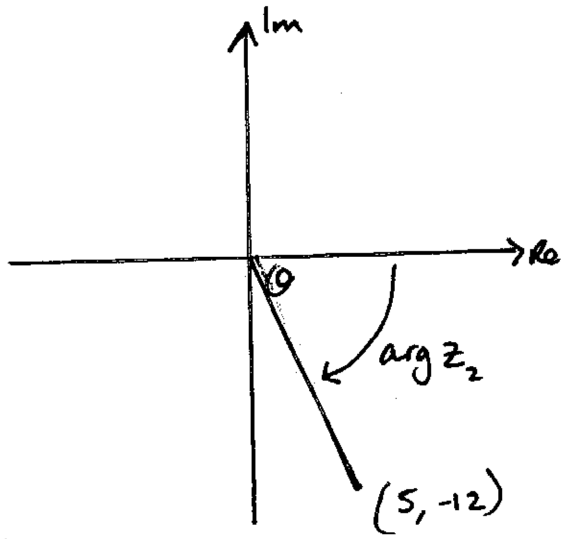
$\theta = \tan^{-1}\left(\frac{4}{3}\right) \simeq 53^\circ$

$\arg z_1 = 180 + 53 = 233^\circ$

or $\arg z_1 = -127^\circ$

Which is the principal argument? The principal argument is -127°

Plot $Z_2 = 5 - 12i$ and find $|Z_2|$ and arg Z_2.



$|z_3| = \sqrt{5^2+12^2} = 13$

$\theta = \tan^{-1}\left(\frac{12}{5}\right) \simeq 67^\circ$

$\arg z_2 = -67^\circ$

Algebra: Complex Numbers

Find $|Z_1Z_2|$ and arg (Z_1Z_2).

$Z_1Z_2 = (-3-4i)(5-12i) = -63+16i \qquad |Z_1Z_2| = \sqrt{63^2+16^2} = 65$

$\arg Z_1Z_2 = 166°$

Conjecture a relationship between $|Z_1|, |Z_2|$ *and* $|Z_1Z_2|$ and between arg Z_1, arg Z_2 and arg(Z_1Z_2).

Conjecture: $|Z_1Z_2| = |Z_1| \times |Z_2|$ since $5 \times 13 = 65$

$\arg(Z_1Z_2) = \arg Z_1 + \arg Z_2$ since $-127° + -67° = -194°$

which is equivalent to 166°

Proof

Let $Z_1 = r_1(\cos A + i\sin A)$ and $Z_2 = r_2(\cos B + i\sin B)$

$Z_1Z_2 = r_1 r_2 (\cos A + i\sin A)(\cos B + i\sin B)$

$= r_1 r_2 (\cos A\cos B + i\sin B\cos A + i\sin A\cos B + i^2\sin A\sin B)$

$= r_1 r_2 (\cos A\cos B - \sin A\sin B + i(\sin A\cos B + \sin B\cos A)$

$= r_1 r_2 (\cos(A+B) + i\sin(A+B))$

$|Z_1Z_2| = r_1 r_2 \qquad \arg(Z_1Z_2) = A+B$

This result extends to division:
$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$ and arg (Z_1Z_2) = arg Z_1 – arg Z_2

And to powers:
$|Z^2| = |Z|^2$ *and* $\arg(Z^2) = 2\arg Z$.

p215 Ex 12.5

- Add or subtract 2π to ensure that arguments are in the range $-\pi$ to π
- Use higher results to simplify
- Consider what will happen with higher powers of Z

De Moivre's Theorem

$|Z^2| = |Z|^2$ *and* $\arg(Z^2) = 2\arg Z$
thus if $Z = r(\cos\theta + i\sin\theta)$ then $Z^2 = r^2(\cos 2\theta + i\sin 2\theta)$.

In general $Z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

De Moivre's Theorem $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

Use proof by induction to show that De Moivre's theorem holds for all positive integers, n.

Let $n=1$ $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ True for $n=1$

Assume true for $n=k$ $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Consider $n=k+1$ $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$ True for $n=k+1$

The theorem is true for $n=1$ and if true for $n=k$, then true for $n=k+1$. Hence by induction, true $\forall n \in \mathbb{N}$.

De Moivre's Theorem in fact holds $\forall\, n \in \boldsymbol{R}$ but proof for non-integer values is beyond the scope of the course.

p218 Ex 12.6 Q 1, 2

Learning to apply De Moivre's Theorem

Example
Express Cos(4θ) as a polynomial in Cos θ

$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$$

$$= \cos^4\theta + 4\cos^3\theta\sin\theta\, i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta\, i + \sin^4\theta$$

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$

Equating real parts:

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta = \cos 4\theta$$

$$\cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 = \cos 4\theta$$

$$\cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta = \cos 4\theta$$

p218 Ex 12.6 Q4, 5, 6, 7
2005 Q12

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

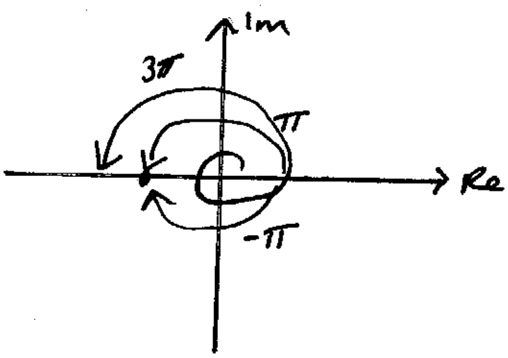
Algebra: Complex Numbers

Learning to find the n^{th} roots of a complex number

- Plot on an argand diagram and write number in polar form
- n equivalent expressions for the argument
- Apply De Moivre's Theorem
- Roots will be equally spaced around the argand diagram (radius = modulus)

Example
Find the cube roots of -1.

Let $Z^3 = -1$ so $Z = (-1)^{1/3}$

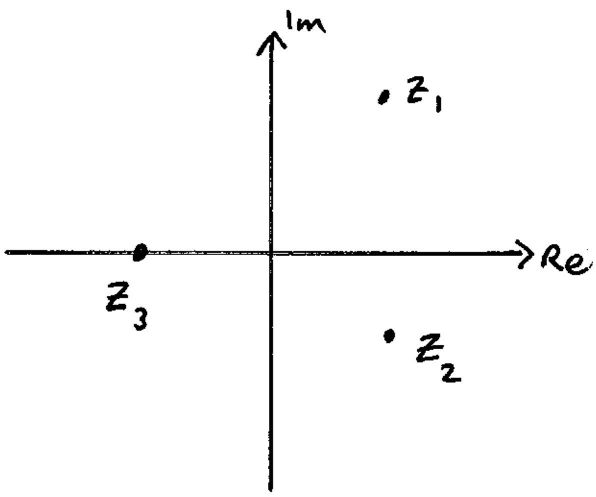


$|Z^3| = 1$ $\qquad$ $\arg Z^3 = \pi \quad = -\pi \quad = 3\pi$

$Z^3 = \cos\pi + i\sin\pi \quad = \cos(-\pi) + i\sin(-\pi) \quad = \cos 3\pi + i\sin 3\pi$

$Z = (\cos\pi + i\sin\pi)^{1/3} \quad = (\cos(-\pi) + i\sin(-\pi))^{1/3} \quad = (\cos 3\pi + i\sin 3\pi)^{1/3}$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \quad = \cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3} \quad = \cos\pi + i\sin\pi$



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Algebra: Complex Numbers

Learning to identify the locus of a point in the complex plane

- Draw a locus using geometrically
- Identify a locus using an algebraic approach

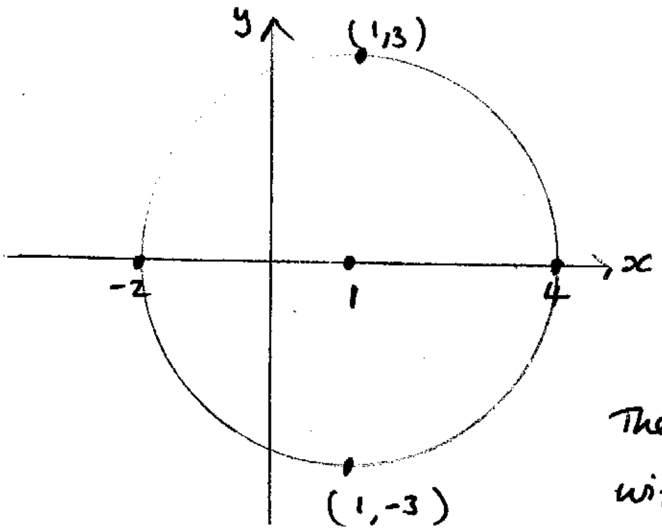
Example 1

Identify the locus in the complex plane given by

$$|Z - 1| = 3$$

Let $z = x + yi$

$|Z - 1|$ indicates the distance from the complex number Z to the number 1 and this has to be always equal to 3, *i.e.* we have a circle, centred on 1 with radius 3.

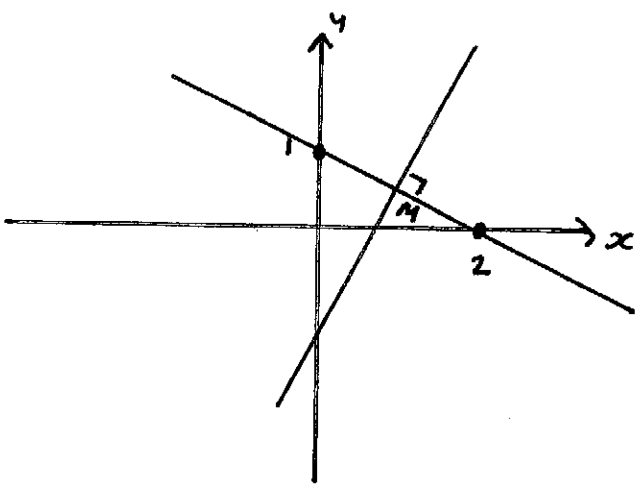


The locus is a circle with equation $(x-1)^2 + y^2 = 9$

Example 2

$|Z - 2| = |Z - i|$

This time the distance from Z to 2 is the same as the distance from Z to *i*. Where are these points?



The required locus is the perpendicular bisector of the line joining (2,0) and (0,1).

Use higher work to determine the equation of the locus.

$M = (1, \frac{1}{2})$

$m_1 = -\frac{1}{2}$ $\quad m_{\perp r} = 2$

The equation of the locus is $y - \frac{1}{2} = 2(x-1)$

$$2y - 1 = 4(x-1)$$

$$2y - 4x + 3 = 0$$

The locus is a straight line with equation $2y - 4x + 3 = 0$

This can also be done by solving the given equation algebraically.

$|Z_1 - Z_2|$ is the distance between two points in the complex plane $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ so $|Z_1 - Z_2|$ can be found using the distance formula:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$|Z - 2| = |Z - i|$

$\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-0)^2 + (y-1)^2}$

$$(x-2)^2 + y^2 = x^2 + (y-1)^2$$

$$x^2 - 4x + 4 + y^2 = x^2 + y^2 - 2y + 1$$

$$-4x + 4 = -2y + 1$$

$$2y - 4x + 3 = 0$$

p 213 Ex 12.4 Q1, 3 and 4