Direct and Indirect Proof

Learning the language of proof

- Know what is meant by an implication
- Write the negation of a statement
- Given an implication, state the **inverse**, the **converse** and the **contrapositive**
- Know that a statement and its contrapositive are logically equivalent

A statement is a sentence that is true or false but not both. For example:

$3 + 5 = 9$ (false!).

A statement is **negated** by putting 'not' after the verb:

$3 + 5 \neq 9$ (true).

A (universal) statement can be disproved by providing one counter example.
Example $x^2 + 1$ is odd $\forall x \in \boldsymbol{R}$
Let $x = 3$, $x^2 + 1 = 10$ which is even. The statement is disproved.
Proving a statement is not quite so easy!

Implications
Consider Pythagoras' Theorem which is an '**if… then**' implication:

If triangle ABC is right-angled at C **then** $c^2 = a^2 + b^2$. Or in symbols:

$$A\hat{C}B = 90° \Rightarrow c^2 = a^2 + b^2$$

The inverse negates both statements: $A\hat{C}B \neq 90° \Rightarrow c^2 \neq a^2 + b^2$

The converse reverses the implication: $c^2 = a^2 + b^2 \Rightarrow A\hat{C}B = 90°$

The contrapositive does both *i.e.* it is the inverse of the converse: $c^2 \neq a^2 + b^2 \Rightarrow A\hat{C}B \neq 90°$

Truth Tables

A	B	Not A	Not B
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

If A then B (implication)	If not A then not B (inverse)	If B then A (converse)	If not B then not A (contrapositive)
T	T	T	T
F	T	T	F
T	F	F	T
T	T	T	T

The truth table for the implication and for the contrapositive are identical and thus they are logically equivalent. If one is proved the other is automatically proven as well. The truth table for the inverse and converse are different from that for the implication so these require further proof.
p3 Ex1.1

Learning to construct some direct proofs

- Prove sums of arithmetic or geometric series
- Prove simple statements involving natural numbers

Examples

1. Prove that the sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.

$S_n = 1 + 2 + 3 + 4 + \cdots + n$

$S_n = n + (n-1) + (n-2) + (n-3) + \cdots + 1$

Adding:

$2S_n = n \times (n+1)$

$S_n = \frac{1}{2}n(n+1)$ as required.

2. Prove that the sum of the first n powers of 2 is $2(2^n-1)$.

$S_n = 2^1 + 2^2 + 2^3 + \cdots + 2^n$

Multiply through by 2:

$2S_n = 2^2 + 2^3 + 2^4 + \cdots + 2^{n+1}$

$2S_n - S_n = 2^{n+1} - 2^1$

$S_n = 2(2^n - 1)$ as required.

3. Prove that $2n^2 + 6n$ is divisible by 4 for all natural numbers, n.

Consider n even and let $n = 2k$ $\quad k \in \mathbb{Z}$

$2n^2 + 6n = 2(2k)^2 + 6(2k)$

$= 8k^2 + 12k$

$= 4(2k^2 + 3k)$ which is divisible by 4.

Consider n odd and let $n = 2k+1$ $\quad k \in \mathbb{Z}$

$2n^2 + 6n = 2(2k+1)^2 + 6(2k+1)$

$= 8k^2 + 8k + 2 + 12k + 6$

$= 8k^2 + 20k + 8$

$= 4(2k^2 + 5k + 2)$ which is divisible by 4.

So n is divisible by 4 $\forall n \in \mathbb{N}$.

p10 Ex1.3 Q1-3, 7, 8, 10, 14, 16 (Ex1.2 for fun!)

Learning indirect proof

- Prove a conjecture by contradiction
- Prove a conjecture by proving the contrapositive

Examples

1. Prove that √2 is irrational.

Assume $\sqrt{2}$ is rational i.e. $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$ with no common factors.

$$b\sqrt{2} = a$$
$$2b^2 = a^2$$

a^2 is even and so a is even

Let $a = 2k \quad k \in \mathbb{Z}$

$$a^2 = 4k^2$$
$$2b^2 = 4k^2$$
$$b^2 = 2k^2$$

b^2 is even and so b is even.

a and b have a common factor of 2, a contradiction so $\sqrt{2}$ is irrational.

2. Prove that there are infinitely many primes.

Assume there is a finite list of prime $p_1, p_2, p_3 \ldots$ with largest prime, p_n.

Let q be a natural number such that

$$q = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1$$

q is not divisible by any of $p_1, p_2, p_3 \ldots, p_n$ (remainder on division is 1) so q is prime.

But, $q > p_n$ a contradiction. Hence, there are infinitely many primes.

3. Prove, by contrapositive, that if $n^2 + 1$ is even then n is odd.

n^2+1 even $\Rightarrow$ n odd

Contrapositive is n not odd $\Rightarrow$ n^2+1 not even

or n even $\Rightarrow$ n^2+1 odd

Let $n = 2k \quad k \in \mathbb{Z}$.

$$n^2+1 = (2k)^2+1$$
$$= 2(2k^2)+1$$ which is one more than a multiple of 2 i.e. odd.

The contrapositive is true and so the original statement is true i.e. if n^2+1 is even then n is odd.

p14 Q1-2, 6-9, 12-14