# Revision of Higher Work

- Know the derivatives of basic functions
- Re-write functions in a form suitable for differentiation
- Use the chain rule to differentiate composite functions, including more than one application

**Standard Derivatives**

| | |
|---|---|
| $f(x) = ax^n$ | $f'(x) = anx^{n-1}$ |
| $f(x) = \sin ax$ | $f'(x) = a\cos ax$ |
| $f(x) = \cos ax$ | $f'(x) = -a\sin ax$ |

**Differentiating sums**

If *f* and *g* are both differentiable then $\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big)$

**Examples**

| | |
|---|---|
| $f(x) = x^3 - 4x + 6 - \frac{4}{x} + \frac{1}{x^3}$ <br> $= x^3 - 4x + 6 - 4x^{-1} + x^{-3}$ | $f'(x) = 3x^2 - 4 + 4x^{-2} - 3x^{-4}$ |
| $f(x) = \sin 2x + \cos 3x$ | $f'(x) = 2\cos 2x - 3\sin 3x$ |
| $f(x) = \frac{x^4 - 2x^2}{x^3}$ <br> $= x - 2x^{-1}$ | $f'(x) = 1 + 2x^{-2}$ |

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## Chain Rule

If $y(x) = f(g(x))$, provided *g* is differentiable at *x* and *f* is differentiable at *g*(*x*), then

$$y'(x) = f'(g(x)) \times g'(x)$$

**Examples**

| | |
|---|---|
| $f(x) = (x^2 + 2x + 3)^5$ | $f'(x) = 5(x^2 + 2x + 3)^4.(2x + 2)$ <br> $= 10(x + 1)\,(x^2 + 2x + 3)^4$ |
| $f(x) = \sin(x^2 - 3)$ | $f'(x) = \cos(x^2 - 3).2x$ |

An informal aide-memoir might be useful:

$$y'(x) = f'(g(x)) \times g'(x)$$

"differentiate w.r.t the bracket and multiply by what you get if you differentiate the bracket".

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## More than one application

In Leibniz notation the chain rule can be written as

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

(This will be important later when we consider related rates of change.)

This 'chain' of derivatives can be extended and allows us to deal with more complex functions. *E.g.* $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$.

## Example

$y = \cos^2(3x)$ *i.e.* $y = (\cos(3x))^2 = u^2$ where $u = \cos(3x) = \cos t$ where $t = 3x$

$$\frac{dt}{dx} = 3, \quad \frac{du}{dt} = -sint, \quad \frac{dy}{du} = 2u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx} \\ &= 2u \times -sint \times 3 \\ &= 2\cos(3x) \times -\sin(3x) \times 3 \\ &= -6\sin(3x)\cos(3x)\end{aligned}$$

This expression can be simplified using the double angle formula for sin
*i.e.* $2\sin A \cos A = \sin 2A$

$-6\sin(3x)\cos(3x) = -3(2\sin(3x)\cos(3x)) = -3\sin 6x$

Final answer $\frac{dy}{dx} = -3\sin 6x$.

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## Extending differentiation to a greater range of functions

- Differentiate a product
- Differentiate a quotient
- Know the derivatives of $e^x$ and $ln\ x$
- Differentiate complex functions using a combination of rules

## The Product Rule

If $y(x) = f(x).g(x)$ then

$$\boxed{\boldsymbol{y'(x) = f'(x)g(x) + f(x)g'(x)}}$$

"Differentiate the first and leave the second alone then leave the first alone and differentiate the second."

**Examples**

1. $y = x^3 \sin 2x$

   $f(x) = x^3$ and $g(x) = \sin 2x$
   $f'(x) = 3x^2$ and $g'(x) = 2\cos 2x$

$$\begin{aligned} y'(x) &= 3x^2 \sin 2x + x^3 . 2\cos 2x \\ &= 3x^2 \sin 2x + 2x^3 \cos 2x \end{aligned}$$

2. $y = (x+3)^3(x^2+2)^2$

   $f(x) = (x+3)^3$ and $g(x) = (x^2+2)^2$
   $f'(x) = 3(x+3)^2.1$ and $g'(x) = 2(x^2+2).2x$

Don't forget chain rule!

Take out common factors

$$\begin{aligned} y'(x) &= 3(x+3)^2(x^2+2)^2 + (x+3)^3.4x(x^2+2) \\ &= (x+3)^2(x^2+2)[3(x^2+2) + 4x(x+3)] \\ &= (x+3)^2(x^2+2)(7x^2+12x+6) \end{aligned}$$

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**Calculus**: Methods in Differentiation

## The Quotient Rule

If $y(x) = \frac{f(x)}{g(x)}$ then $\boxed{\boldsymbol{y'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}}}$

**Examples**

1. $y = \frac{x^2-2x+1}{x^2+1}$

$f(x) = x^2 - 2x + 1$ and $g(x) = x^2 + 1$
$f'(x) = 2x - 2$ $\quad g'(x) = 2x$

$$y'(x) = \frac{(2x-2)(x^2+1) - (x^2-2x+1)(2x)}{(x^2+1)^2}$$
$$= \frac{2x^3-2x^2+2x-2-2x^3+4x^2-2x}{(x^2+1)^2}$$
$$= \frac{2x^2-2}{(x^2+1)^2}$$

2. $y = \tan x = \frac{\sin x}{\cos x}$

$f(x) = \sin x$ and $g(x) = \cos x$

$f'(x) = \cos x$ $\quad g'(x) = -sinx$

$$y'(x) = \frac{\cos x.\cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$

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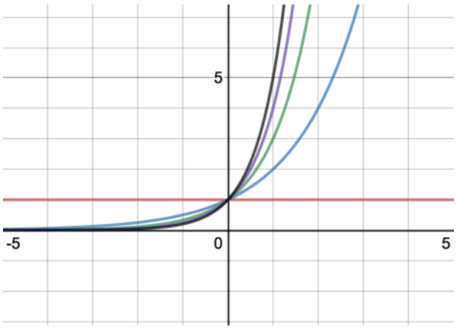
## Derivatives of exponential and logarithmic functions




A function of the form $y = a^x$ is an exponential function. The graph shows a selection of exponential functions with different values of a. All the curves pass through (0,1) but each curve has a different gradient ranging from 0 when $a$ = 1 to infinity as $a$ increases. A gradient of 1 at (0,1) occurs for a value of $a$ between 2 and 3. This base is defined as the number $e \approx 2.718281$ ...

### Differentiating an exponential function

Let $f(x) = a^x$

From the definition of a derivative $f'(x) = \lim_{h\to 0}\left(\frac{a^{x+h}-a^x}{h}\right)$

$$f'(x) = \lim_{h\to 0}\left(\frac{a^x \times a^h - a^x}{h}\right)$$

Using laws of indices

$$f'(x) = \lim_{h\to 0}\left(\frac{a^x(a^h - 1)}{h}\right)$$

$$f'(x) = \lim_{h\to 0} a^x \times \lim_{h\to 0}\left(\frac{a^h - 1}{h}\right)$$

$$f'(x) = a^x \times gradient\ at\ (0,1)$$

For $f(x) = e^x$ gradient at (0,1) is 1 so $f'(x) = e^x$ !

$$\boxed{\begin{aligned} \boldsymbol{f(x)} &= \boldsymbol{e^x} \\ \boldsymbol{f'(x)} &= \boldsymbol{e^x} \end{aligned}}$$

**Example**

$f(x) = e^{2x}$
$f'(x) = e^{2x} \times 2$
$f'(x) = 2e^{2x}$

chain rule

**Calculus**: Methods in Differentiation

**Differentiating the logarithmic function**

The natural logarithm is the inverse of the exponential function, base $e$.

Notation:
The natural logarithm can be written as $\log_e x$ but this is usually abbreviated to $\ln x$.

If $y = e^x$ then $\ln y = x$ and $e^{\ln x} = x$

$$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x$$
$$e^{lnx} \times \frac{d}{dx}\ln x = 1$$
$$x \times \frac{d}{dx}\ln x = 1$$
$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

**Example**

$$y = \frac{\ln x}{x^2}$$

This is a quotient with $f(x) = \ln x$ and $g(x) = x^2$

$f'(x) = \frac{1}{x}$ and $g'(x) = 2x$

$$\frac{dy}{dx} = \frac{\frac{1}{x}.x^2 - \ln x.2x}{(x^2)^2}$$
$$= \frac{x - 2x\ln x}{x^4}$$
$$= \frac{x(1 - 2\ln x)}{x^4}$$
$$= \frac{1 - 2\ln x}{x^3}$$

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