1.4 Applying Algebraic skills to Number Theory

Learning to apply the division algorithm

- Know the division algorithm
- Apply the division algorithm repeatedly to find the *gcd* of two positive integers
- Express the *gcd* as a linear combination of the original two numbers
- Express a base 10 number in another number base

The Division Algorithm

$170 \div 7 = 24 \; r \; 2$ or $170 = 24 \times 7 + 2$

In fact, for any positive integers a and b there exist unique integers q and r $0 \le r \le b$ such that

$$\boxed{a = qb + r}$$

This is known as the **division algorithm.**

Greatest Common Divisor
In the context of number theory, and working with positive integers, (a,b) denotes the *gcd* of a and b.

Theorem If $a = qb + r$ then $(a,b) = (b,r)$

Proof

Let $d = (a, b)$ $d|a$ and $d|b$ "d divides a"

Since $a = qb+r$ then $d|r$ so d is a common diviser of a, b and r

Let $e = (b, r)$ $e|b$ and $e|r$

Since $a = qb+r$, $e|a$ so e is a common diviser of a and b

But d is the greatest common diviser of a and b so $e \le d$

We defined e to be the greatest common diviser of b and r

So there cannot be a bigger diviser. Thus, $e = d$ and $d = (b, r)$

Repeated application of the above theorem to find the *gcd* of two positive integers is known as the **Euclidian Algorithm.**

Example

Find the *gcd* of 408 and 153

(1) $408 = 2 \times 153 + 102$ $\qquad (408, 153) = (153, 102)$

(2) $153 = 1 \times 102 + 51$ $\qquad (153, 102) = (102, 51)$

(3) $102 = 2 \times 51 + 0$ $\qquad (102, 51) = (51, 0)$

gcd of 408 and 153 is 51.

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Continuing the above Example

Express 51 as a linear combination of 408 and 153

In (1) $408 = 2 \times 153 + 102 \Rightarrow 102 = 408 - 2 \times 153$

(2) $153 = 1 \times 102 + 51 \Rightarrow 51 = 153 - 102$

Substituting (1) in (2)

$51 = 153 - (408 - 2 \times 153)$

$51 = 3 \times 153 - 408$

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Number Bases

To express a base 10 number in another number base we can use repeated division.

Examples

1. Express 1342_{10} in base 8.

$1342 \div 8 = 167 \text{ r } 6$

$167 \div 8 = 20 \text{ r } 7$

$20 \div 8 = 2 \text{ r } 4$

$2 \div 8 = 0 \text{ r } 2$

Reading the remainders upwards we get: $2476_8 = 1342_{10}$

check: $\overset{8^3}{2}\overset{8^2}{4}\overset{8^1}{7}\overset{8^0}{6}$ $\quad 2 \times 8^3 + 4 \times 8^2 + 7 \times 8 + 6 = 1342.$

2. Express 2305_7 in base 8

First change 2305_7 to a base 10 number:

$2 \times 7^3 + 3 \times 7^2 + 0 + 5 = 838$

$838 \div 8 = 104 \text{ r } 6$

$104 \div 8 = 13 \text{ r } 0$

$13 \div 8 = 1 \text{ r } 5$

$1 \div 8 = 0 \text{ r } 1$

$2305_7 = 838_{10} = 1506_8$

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