Partial Fractions

Learning to express an algebraic fraction as a sum of partial fractions

- Identify linear factors
- Obtain an *identity* by adding appropriate fractions
- Use identity to determine constants

We can express $\frac{2}{3x+1}+\frac{5}{x-2}$ as a single fraction:

$$\frac{2}{3x+1}+\frac{5}{x-2}=\frac{2(x-2)}{(3x+1)(x-2)}+\frac{5(3x+1)}{(3x+1)(x-2)}=\frac{17x+1}{(3x+1)(x-2)}$$

Sometimes we may want to work backwards and express $\frac{17x+1}{(3x+1)(x-2)}$ as a sum of simpler fractions. We assume:

$\frac{17x+1}{(3x+1)(x-2)} \equiv \frac{A}{3x+1}+\frac{B}{x-2}$, where A and B and numbers to be determined.

$$\frac{17x+1}{(3x+1)(x-2)} \equiv \frac{A(x-2)+B(3x+1)}{(3x+1)(x-2)}$$

$$17x+1 \equiv A(x-2)+B(3x+1)$$

Let $x=2$ $\quad 35=7B$

$B=5$

Let $x=0$ $\quad 1=-2A+5$

$A=2$

$$\frac{17x+1}{(3x+1)(x-2)} \equiv \frac{2}{3x+1}+\frac{5}{x-2}$$

> $\equiv$ means 'identically equal to'. An identity is true **for all** values of *x*. Thus we can substitute any convenient values of *x* to determine *A* and *B*.

Algebra Partial Factions

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Learning to identify the various forms of partial fractions

- Know the form of partial fractions for a repeated linear factor
- Know the form of partial fractions for an irreducible quadratic factor

Example 1

Express in partial fractions $\frac{x}{(x+1)^2(x-2)}$

$$\frac{x}{(x+1)^2(x-2)} \equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\equiv \frac{A(x+1)^2 + B(x-2)(x+1) + C(x-2)}{(x+1)^2(x-2)}$$

$$x \equiv A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

Let $x = -1$ $\quad -1 = -3C \quad C = 1/3$

$x = 2 \quad 2 = 9A \quad A = 2/9$

$x = 0 \quad 0 = \frac{2}{9} - 2B - \frac{2}{3} \quad B = -2/9$

$$\frac{x}{(x+1)^2(x-2)} \equiv \frac{2}{9(x-2)} - \frac{2}{9(x+1)} + \frac{1}{3(x+1)^2}$$

Example 2

Express in partial fractions $\frac{x^2+5x+3}{(x-1)\ (x^2+2)}$

$$\frac{x^2+5x+3}{(x-1)(x^2+2)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

$$\equiv \frac{A(x^2+2) + (Bx+C)(x-1)}{(x-1)(x^2+2)}$$

$$x^2 + 5x + 3 \equiv A(x^2+2) + (Bx+C)(x-1)$$

Let $x = 1 \quad 9 = 3A \quad A = 3$

Let $x = 0 \quad 3 = 6 - C \quad C = 3$

Let $x = -1 \quad -1 = 9 + 2B - 6 \quad B = -2$

$$\frac{x^2+5x+3}{(x-1)(x^2+2)} \equiv \frac{3}{x-1} + \frac{3-2x}{x^2+2}$$

Algebra Partial Factions

Summary

Denominator	Example	Form of Expression
Linear factors	$\dfrac{5}{(x-2)(x+3)}$	$\dfrac{A}{x-2}+\dfrac{B}{x+3}$
Repeated factor	$\dfrac{5}{(x-2)(x+3)^2}$	$\dfrac{A}{x-2}+\dfrac{B}{x+3}+\dfrac{C}{x+3}$
Quadratic factor	$\dfrac{2x+3}{(x-1)(x^2+4)}$	$\dfrac{A}{x-1}+\dfrac{Bx+c}{(x^2+4)}$

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Learning to express an improper fraction as a polynomial and a sum of partial fractions

- Recognise improper algebraic fractions
- Perform polynomial long division
- Apply method to express fraction as sum of partial fractions

Recall long division of numbers!
e.g. $4628 \div 17$ $= 272 + \frac{4}{17}$

$$\begin{array}{r}
272 \\
17\,\overline{)\,4628} \\
-34 \\
\hline
122 \\
119 \\
\hline
38 \\
34 \\
\hline
4
\end{array}$$

Example 1
Divide $(x^3-1) \div (x^2+1)$ $= x - \dfrac{x+1}{x^2+1}$

$$\begin{array}{r}
x \\
x^2+1\,\overline{)\,x^3+0x^2+0x-1} \\
-\ x^3+0+x \\
\hline
-x-1
\end{array}$$

Example 2

Express as a polynomial and a sum of partial fractions $\frac{x^3}{(x+1)(x-3)}$.

$$\begin{array}{r} x + 2 \\ x^2 - 2x - 3 \overline{)\, x^3 \qquad\qquad\quad} \\ \underline{x^3 - 2x^2 - 3x} \\ 2x^2 + 3x \quad\; \\ \underline{2x^2 - 4x - 6} \\ 7x + 6 \end{array}$$

$$\frac{x^3}{(x+1)(x-3)} = x + 2 + \frac{7x+6}{(x+1)(x-3)}$$

$$\frac{7x+6}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

$$= \frac{A(x-3) + B(x+1)}{(x+1)(x-3)}$$

$$7x + 6 = A(x-3) + B(x+1)$$

Let $x = 3$ $\quad 27 = 4B$

$$B = \frac{27}{4}$$

Let $x = -1$ $\quad -1 = -4A$

$$A = \frac{1}{4}$$

$$\frac{x^3}{(x+1)(x-3)} = x + 2 + \frac{1}{4(x+1)} + \frac{27}{4(x-3)}$$