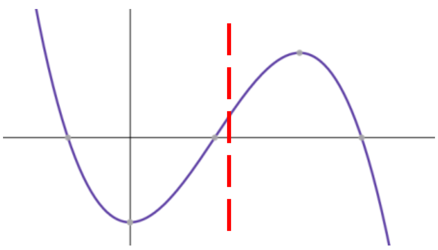
Lesson 1 Domain and range of functions and inverse functions

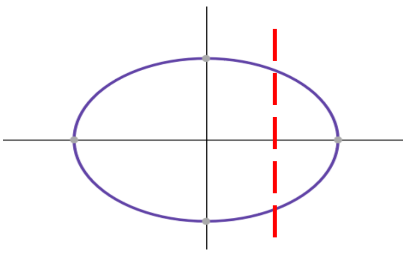
A function is a rule which assigns each member of the domain (the set of all values of x) to one and only one member of the range (the set of all values of $f(x)$ or y).

Graphically this can be shown by drawing a vertical line through any point in the domain. If the line and the curve have only one point, then the function is one-to-one. Two or more points of intersection - not a function

For example



A cubic function



An ellipse – not one-to-one

In order to preserve this one-to-one correspondence some commonly used functions have restrictions on their domain.

The square root function $y = \sqrt{x}$ has the domain $x \in \mathbb{R}^+$ and the range $f(x) \in \mathbb{R}^+$

A rational function such as $y = \frac{1}{x-3}$ has the domain $x \in \mathbb{R} - \{3\}$ and range $f(x) \in \mathbb{R} - \{0\}$

For $y = \tan x$ the asymptotes $x = \pm \frac{n\pi}{2}$ restrict the domain to $\mathbb{R} - \{\frac{n\pi}{2}\}$

Quadratic functions such as $y = x^2 + 4x + 5$ have a domain $\mathbb{R}$ and range $f(x) \geq 1$

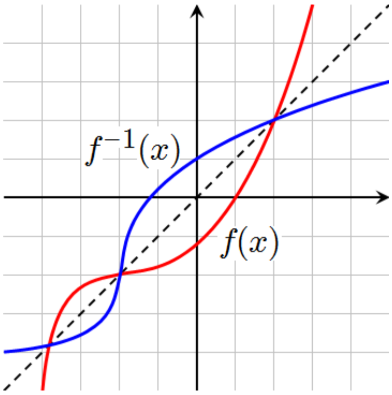
Unless otherwise stated the domain and range of a function is assumed to be $\mathbb{R}$ the set of real numbers.

In MIA textbook - 5.1 Q1 and 3

Lesson 1 Inverse functions

Where there exists a one-to-one correspondence between the domain and the range, then there also exists an inverse function which links each element in $f(x)$ back to x. If a function has an inverse, then it is "invertible".

This can be found graphically by reflecting the function in the line $y = x$



An inverse function can also be found algebraically by changing the subject of the formula. Remember that inverse functions might also have restricted domains

Function $f(x)$	Inverse function $f^{-1}(x)$
$y = x^2 - 4$	$y = \sqrt{x+4}$ domain $\mathbb{R} \geq -4$
$y = \sin x$	$y = \sin^{-1} x$ domain $-1 \leq x \leq 1$
$y = \tan x \quad \mathbb{R} - \left\{\frac{n\pi}{2}\right\}$	$y = \tan^{-1} x$
$y = x^2 + 4x + 5$	$y = \sqrt{x-1} - 2$ domain $\mathbb{R} \geq 1$
$y = \frac{1}{x-3}$ domain $\mathbb{R} - \{3\}$	$y = \frac{1}{x} + 3$ domain $\mathbb{R} - \{0\}$

In MIA textbook - 5.3 Q1 and 3, (Q2 for extension)

In Leckie and Leckie – Exercise 6K Q1 to 3

Lesson 1 The modulus function $f(x) = |x|$

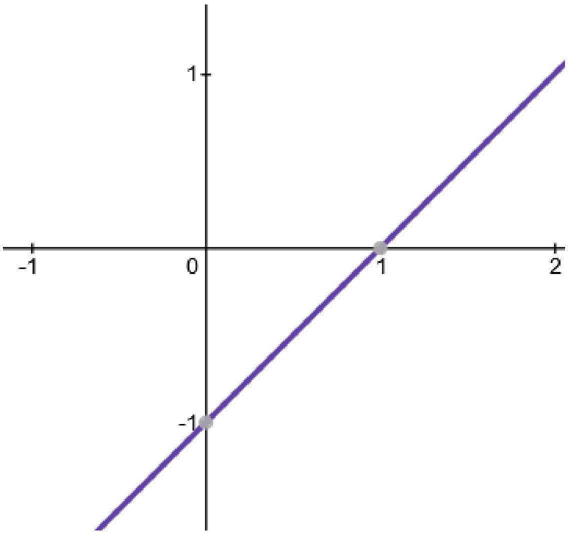
The modulus or absolute value of x is defined as the non-negative value of x without regard for its sign. So $|x| = x$ for a positive x and $|x| = -x$ when x is negative.

The modulus function has some nice properties

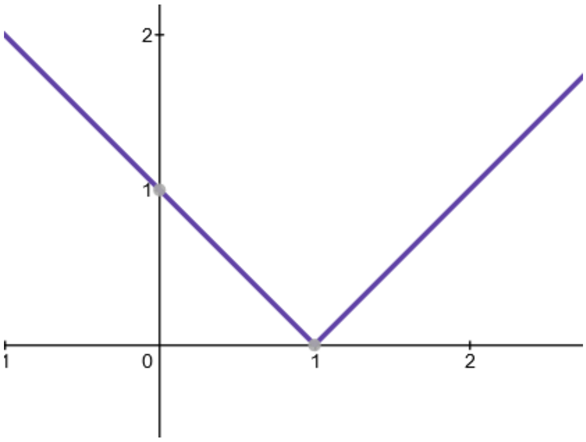
1. $|x| = \sqrt{x^2}$ $\quad |-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$
2. $|x + y| \leq |x| + |y|$ $\quad |-3 + 2| = 1, \quad |-3| + |1| = 4, \ 1 < 4$
3. $|x \times y| = |x| \times |y|$ $\quad |-3 \times 5\,| = 15 = |-3| \times |5|$
4. $|x| \leq a, -a \leq x \leq a$ $\quad |5| < 7, \quad -7 < 5 < 7$
5. $|x| \geq a, -a \geq x \ \ or \ \ x \geq a$ $\quad |-3| > 2, \ -2 > \ -3 \ \text{true}, -3 > 2 \ \text{false}$

When sketching the modulus function, reflect the negative portion of the graph in the x –axis. Always clearly mark any intercepts

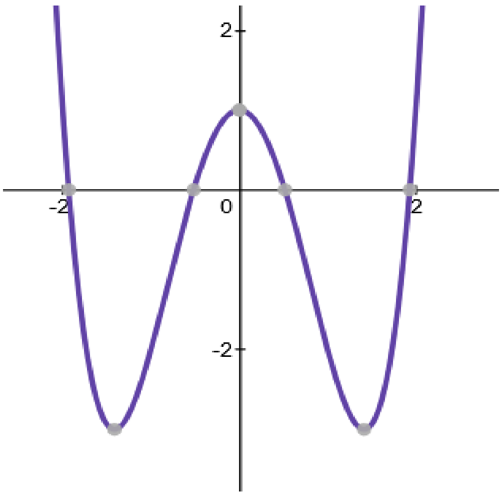
$\boldsymbol{y = x - 1}$,



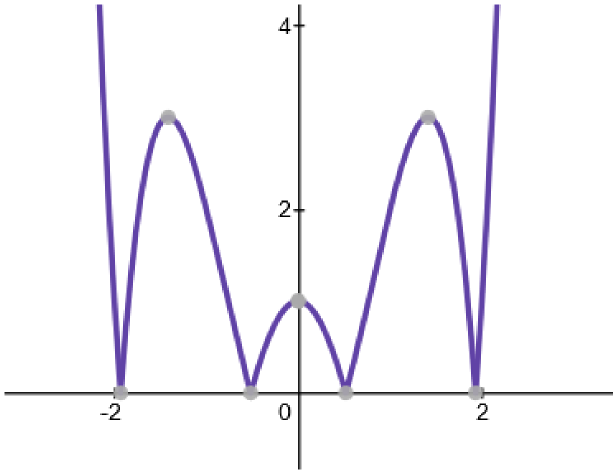
$\boldsymbol{y = |x - 1|}$



$y = f(x)$



$y = |f(x)|$

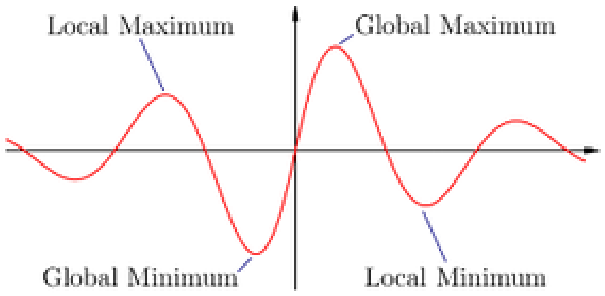


In MIA textbook - 5.2 all of the exercise

In Leckie and Leckie – Exercise 6K Q4 to 7

Lesson 2 Find the extrema (max/min) of functions in a closed interval

A continuous function f can have both local and global maximums and minimums.



A global maximum has the property that $f(\alpha) > f(x)$ for all x
A global minimum has the property that $f(\alpha) < f(x)$ for all x

Local maxima/minima occur at critical points within a closed interval. These can either be endpoints or points where $f'(\alpha) = 0$ or $f'(\alpha)$ is undefined.

To identify these extrema on the interval $a \leq x \leq b$ or $[a, b]$:

1. Use differentiation to identify where $f'(\alpha) = 0$ or $f'(\alpha)$ is undefined
2. Identify the absolute extrema for the function $f(a)$ and $f(b)$
3. State a conclusion

Example 1

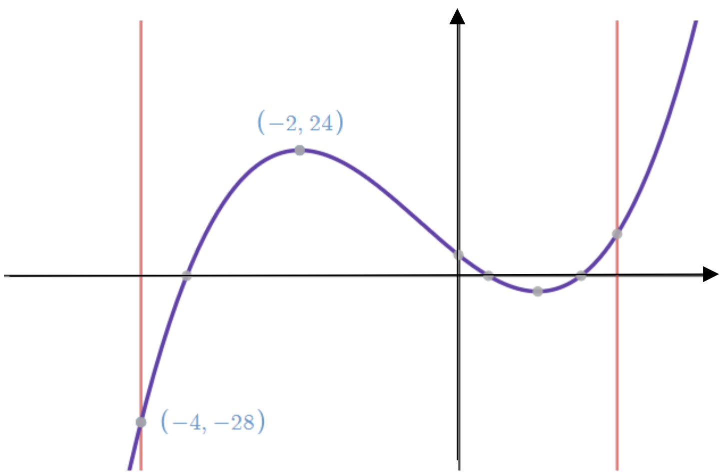
Determine the extrema for $f(x) = 2x^3 + 3x^2 - 12x + 4$ on $[-4,2]$

$f'(x) = 6x^2 + 6x - 12, \quad 6(x-1)(x+2) = 0, \quad x = 1, \ x = -2$

$f(1) = -3, \ f(-2) = 24 \qquad\qquad f(-4) = -28, \ f(2) = 8$

For this interval $(-4, -28)$ is a local minimum, $(-2,24)$ is a local maximum.

This function has no global absolutes.



Example 2 $f(x) = |x^2 - 2x - 8|$ on $[-3,3]$

For a modulus function $|x| = \sqrt{x^2}$ so $y = \sqrt{x^2}$ where $u = x^2 - 2x - 8$

Using the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 1 \times (2x - 2)$ $\frac{dy}{dx} = 2x - 2$

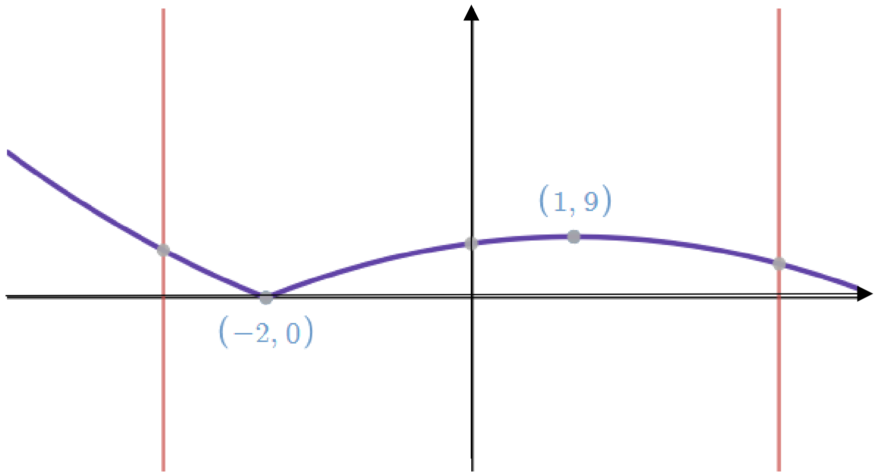
There is a critical point when $x = 1$

From a sketch of the graph there is also a critical point when $x = -2$

$f(1) = 9,\ f(-2) = 0,\ f(-3) = 7, f(3) = 5$

Within this interval $(-2,0)$ is a local minimum, $(1,9)$ is a local maximum.

The point $(-2,0)$ is also a global minimum (together with the point $(4,0)$)

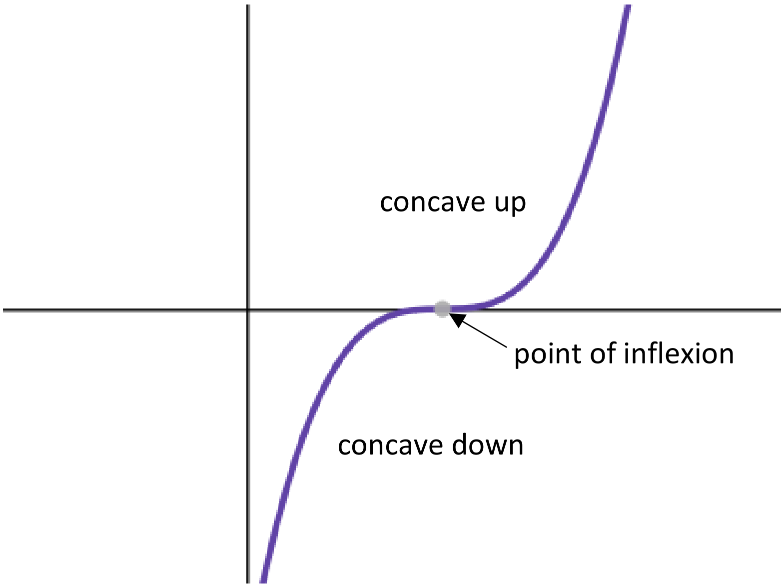


In MIA - Sketch a few graphs from Exercise 5.5 then do Q3 from Exercise 5.6

In Leckie and Leckie – Exercise 6F on page 195, Q1 and 3

Lesson 3 Concavity and points of inflexion

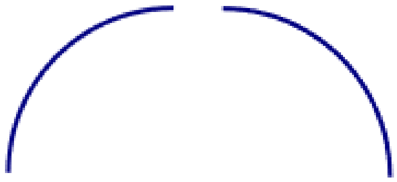
Points of inflexion are critical points where a continuous function changes concavity – changes from concave down to concave up or vice versa.



A function f is concave up if it "opens" up.

Or - the tangents to the curve are all below the curve within this interval.

Or - $f''(x) > 0$ for all x within this interval.



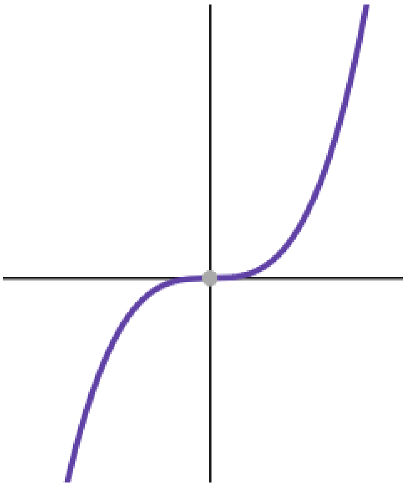
A function f is concave down if it "opens" down.

Or - the tangents to the curve are all above the curve within this interval.

Or - $f''(x) < 0$ for all x within this interval.

Points of inflexion occur when the second derivative is either zero or undefined **and** concavity is different on both sides of the point. The fact that the second derivative is either zero or undefined is not enough to determine a point of inflexion and you should use a table of signs of $f''(x)$ to confirm any changes of concavity.

Example 1



$f(x) = x^5$

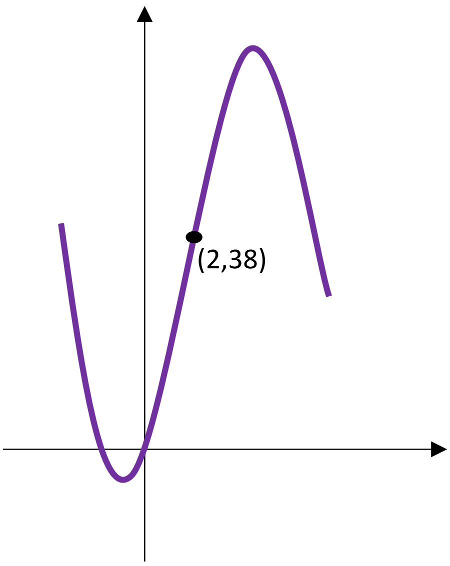
$f'(x) = 5x^4,$

$f''(x) = 20x^3, \quad f''(x) = 0$ when $x = 0$

x	$\rightarrow$	0	$\rightarrow$
$f''(x)$	negative	zero	positive
Concavity	down		up

Change of concavity, hence $(0,0)$ is a point of inflexion

Example 2



$f(x) = 6x^2 + 12x - x^3 - 2$

$f'(x) = 12x + 12 - 3x^2$

$f''(x) = 12 - 6x \quad f''(x) = 0$ when $x = 2$

x	$\rightarrow$	2	$\rightarrow$
$f''(x)$	positive	zero	negative
Concavity	up		down

Change of concavity, hence $(2,38)$ is a point of inflexion

Example 3 *be careful when using $f''(x) = 0$ to identify points of inflexion*

$f(x) = \frac{1}{3}x^6, \quad f'(x) = 2x^5, \quad f''(x) = 10x^4 \quad f''(0) = 0$

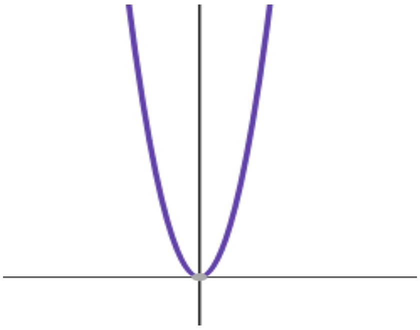
x	$\rightarrow$	0	$\rightarrow$
$f''(x)$	+	0	+
Concavity	up		up

There is no change in concavity,
so $(0,0)$ is not a point of inflexion

In MIA - Exercise 5.7 work through the examples, Q2 and Q3 are sufficient
In Leckie and Leckie: Exercise 6D – stationary points, Exercise 6E – concavity and points of inflexion. (only do a few questions from each exercise)

Lesson 4 Odd and Even functions

An **even** function is symmetric about the y-axis and has the property that $\boldsymbol{f(-x) = f(x)}$



$f(x) = x^2$

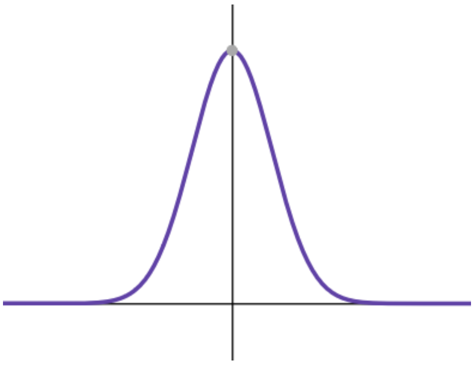
$f(-x) = (-x)^2 = x^2 = f(x)$

$f(-x) = f(x)$ so function f is even

$$g(x) = 10e^{-0.2x^2}$$

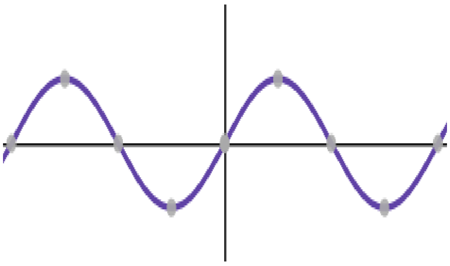
$g(-x) = 10e^{-0.2(-x)^2} = 10e^{-0.2x^2} = g(x)$

$g(-x) = g(x)$ so function g is even



An **odd** function has half-turn symmetry about the origin and has the property

$\boldsymbol{f(-x) = -f(x)}$



$f(x) = \sin x$

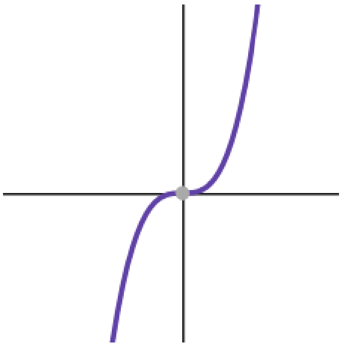
$f(-x) = \sin(-x) = -\sin x = -f(x)$

$f(-x) = -f(x)$ so function f is odd

$h(x) = x^3$

$h(-x) = (-x)^3 = -x^3 = -h(x)$

$h(-x) = -h(x)$ so function h is odd



Example 1 $f(x) = -2x^2 + 4,$

$$f(-x) = -2(-x)^2 + 4 = -2x^2 + 4 = f(x)$$

$f(-x) = f(x)$ so function f is even

Example 2 $g(x) = 3|x| - x^2,$

$$g(-x) = 3|-x| - (-x^2) = 3|x| - x^2 = g(x)$$

$g(-x) = g(x)$ so function g is even

Example 3 $h(x) = \frac{1}{x},$

$$h(-x) = \frac{1}{-x} = -\frac{1}{x} = -h(x),$$

$h(-x) = -h(x)$ so function h is odd

Example 4 $j(x) = x^2 + 3x,$

$$j(-x) = (-x)^2 + 3(-x) = x^2 - 3x,$$

$j(-x) \neq j(x)$, $j(-x) \neq -j(x)$ so function j is neither even nor odd.

In MIA - Exercise 5.8, do the whole exercise

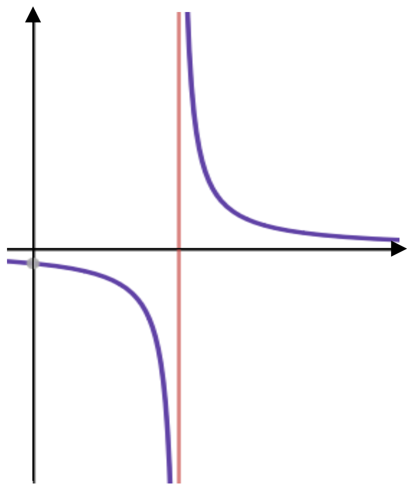
In Leckie and Leckie – Exercise 6G Q1 to 4

Lesson 5 Asymptotes – vertical, horizontal and oblique

Where $f(x)$ is a rational function in the form $f(x) = \frac{g(x)}{h(x)}$

Vertical asymptotes $\boldsymbol{x = a}$ are where the function is undefined or discontinuous. These can be found by solving $h(a) = 0$

Example 1



$$y = \frac{1}{x-3},$$

$h(x) = 0, \quad x - 3 = 0, \quad x = 3$

Thus there is a **vertical asymptote at $\boldsymbol{x = 3}$**

As $x \to 3$, from the left the denominator tends to ∞^{-}.

As $x \to 3$, from the right the denominator tends to ∞^{+}.

Vertical asymptotes indicate very specific behaviour on a graph and are usually found close to the origin. The function will never touch or cross vertical asymptotes.

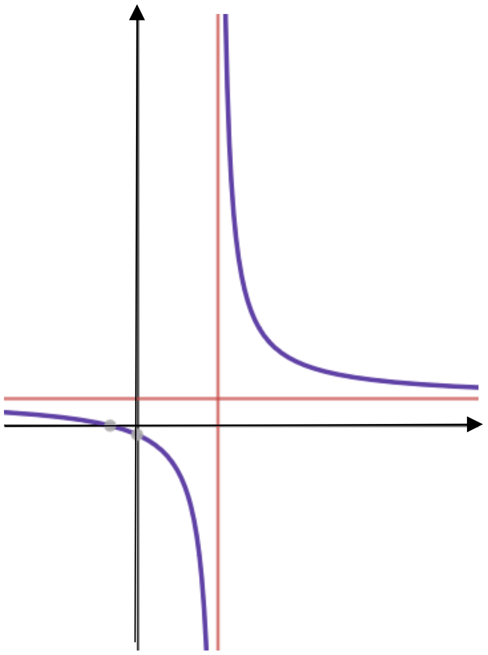
However horizontal asymptotes indicate more general behaviour of a graph and the function can cross these lines. The method used to find horizontal asymptotes changes depending on the degree of the numerator

1. If the numerator $g(x)$ has a lower degree than the denominator $h(x)$ then the **$\boldsymbol{x}$-axis** is the horizontal asymptote.

2. If both polynomials have the same degree use algebraic long division to express the function in the form $f(x) = b + \frac{g(x)}{h(x)}$. The horizontal asymptote is the line $\boldsymbol{y = b}$

3. If the numerator $g(x)$ has a higher degree than the denominator $h(x)$ then there is an oblique asymptote. Express the function in the form $f(x) = x + c + \frac{g(x)}{h(x)}$, to identify the oblique asymptote $\boldsymbol{y = x + c}$. The graph of a function will not touch or cross an oblique asymptote.

Example 2



$$y = \frac{x+1}{x-3},$$

$h(x) = 0, \quad x - 3 = 0, \quad x = 3$

Thus there is a vertical asymptote at $x = 3$

$$y = 1 + \frac{4}{x-3}$$

There is a **horizontal asymptote at $\boldsymbol{y = 1}$**

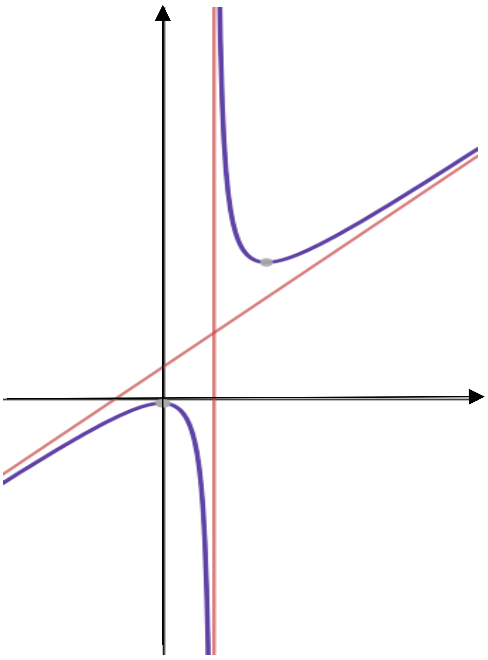
As $x \to \infty^+$, $y \to 1 + 0^+$,

graph approaches $y = 1$ from above.

As $x \to \infty^-$, $y \to 1 + 0^-$

graph approaches $y = 1$ from below.

Example 3



$$y = \frac{x^2+1}{x-3},$$

Vertical asymptote at $x = 3$

$$y = (x+3) + \frac{10}{x-3}$$

There is an **oblique asymptote at $\boldsymbol{y = x + 3}$**

As $x \to \infty^+$, $y \to x + 3 + 0^+$,

graph approaches $y = x + 3$ from above.

As $x \to \infty^-$, $y \to x + 3 + 0^-$

graph approaches $y = x + 3$ from below.

In MIA - Exercise 5.9 Question 1 and Exercise 5.10 Questions 1 and 3
In Leckie and Leckie:

Exercise 6A - vertical asymptotes, Exercise 6B – horizontal asymptotes and Exercise 6C – oblique asymptotes (only do a few questions from each exercise)

Lesson 6 A complete sketch of a rational function

Identify:

1. **Intercepts** $x = 0,\ y = 0$
2. **Stationary points** $f'(x) = 0,$ minimum when $f''(x) > 0,$
   maximum when $f''(x) < 0,$
   points of inflexion $f''(x) = 0$
3. **Asymptotes** vertical when denominator $= 0$
   Non-vertical as $x \to \infty^{\pm}$

Then produce a fully annotated sketch. Remember you can use Desmos and your calculator when working at home, but you might have to do this without these aids in an exam.

Example 1

Sketch $y = \frac{x^2-9}{x^2-4}$

1. When $x = 0,\ y = \frac{9}{4}$ $\left(0, \frac{9}{4}\right)$.
   When $y = 0,\ x^2 - 9 = 0,\ x = \pm 3$ $(-3,0), (3,0)$

2. $y = \frac{x^2-9}{x^2-4},$ $\frac{dy}{dx} = \frac{(2x)(x^2-4)-(2x)(x^2-9)}{(x^2-4)^2} = \frac{10x}{(x^2-4)^2}$
   $\frac{d^2y}{dx^2} = \frac{10(x^2-4)^2-(10x)4x(x^2-4)}{(x^2-4)^4} = \frac{-10(3x^2+4)}{(x^2-4)^3}$

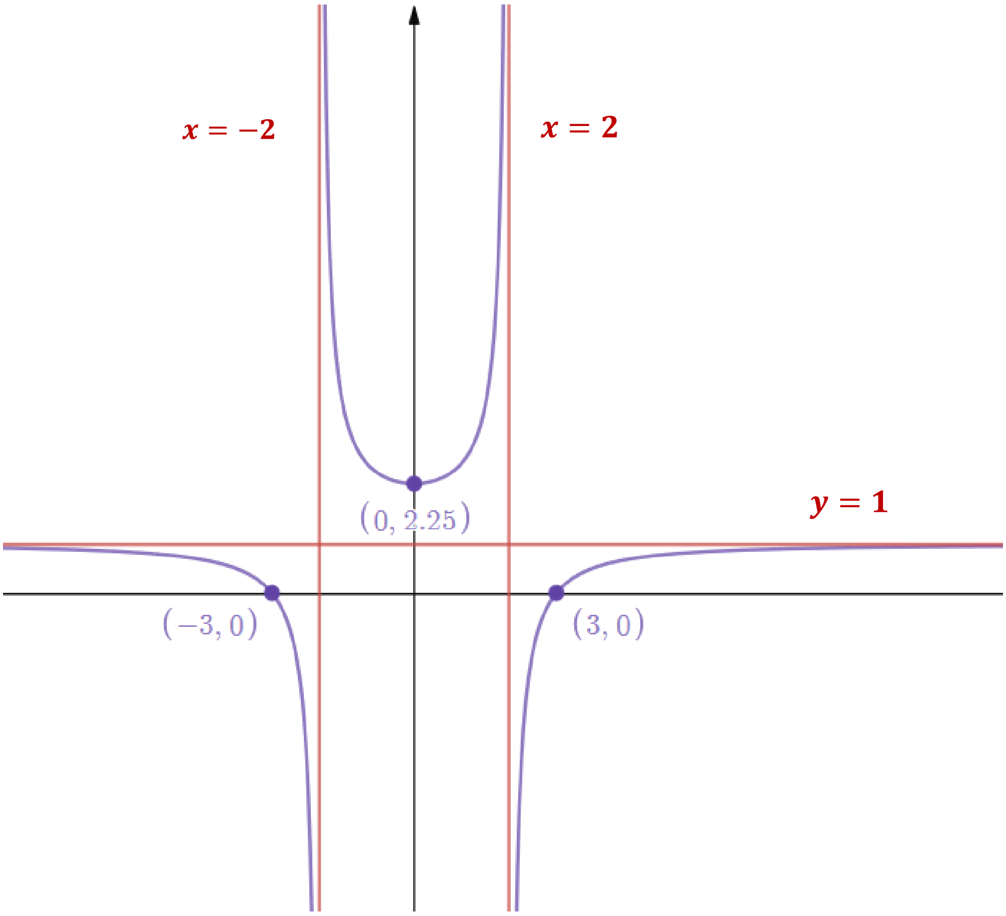
   $\frac{dy}{dx} = 0$ when $x = 0,$ $\frac{d^2y}{dx^2} = \frac{5}{8},$ minimum stationary point at $\left(0, \frac{9}{4}\right),$

   $\frac{d^2y}{dx^2} = 0$ has no real roots, so no points of inflexion.

3. Vertical asymptotes at $x = \pm 2$
   Using algebraic long division $f(x) = \frac{x^2-9}{x^2-4} = 1 - \frac{5}{x^2-4}$
   There is a horizontal asymptote at $y = 1$

   As $x \to \infty^{+},\ y \to 1 - 0^{+}$, graph approaches $y = 1$ from below.

   As $x \to \infty^{-},\ y \to 1 - 0^{-}$ graph approaches $y = 1$ from below.



In MIA textbook - Exercise 5.11 Question 1, 2 and Q5 in Review

In Leckie and Leckie – Exercise 6I on page 206 Q1

Example 2

Sketch $y=\frac{x^3}{1-x^2}$

1. When $x=0,\ y=0\ \ (0,0)$.
   When $y=0,\ x^3=0,\ x=\ 0\ \ (0,0)$.

2. $y=\frac{x^3}{1-x^2}$, $\quad \frac{dy}{dx}=\frac{3x^2(1-x^2)+2x(x^3)}{(1-x^2)^2}=\frac{3x^2-x^4}{(1-x^2)^2}$

$$\frac{d^2y}{dx^2}=\frac{(6x-4x^3)(1-x^2)^2-2(1-x^2)(-2x)(3x^2-x^4)}{(1-x^2)^4}=\frac{6x+2x^3}{(1-x^2)^3}$$

$\frac{dy}{dx}=0,\ \ 3x^2-x^4=0,\ \ x^2(3-x^2),\ \ x=0, x=\ \pm\sqrt{3}$

$f''(0)=0,\ f''(-\sqrt{3})>0\ ,\ f''(\sqrt{3})<0$

minimum stationary point at $(-\sqrt{3}, 2.598)$,

maximum stationary point at $(\sqrt{3}, -2.598)$,

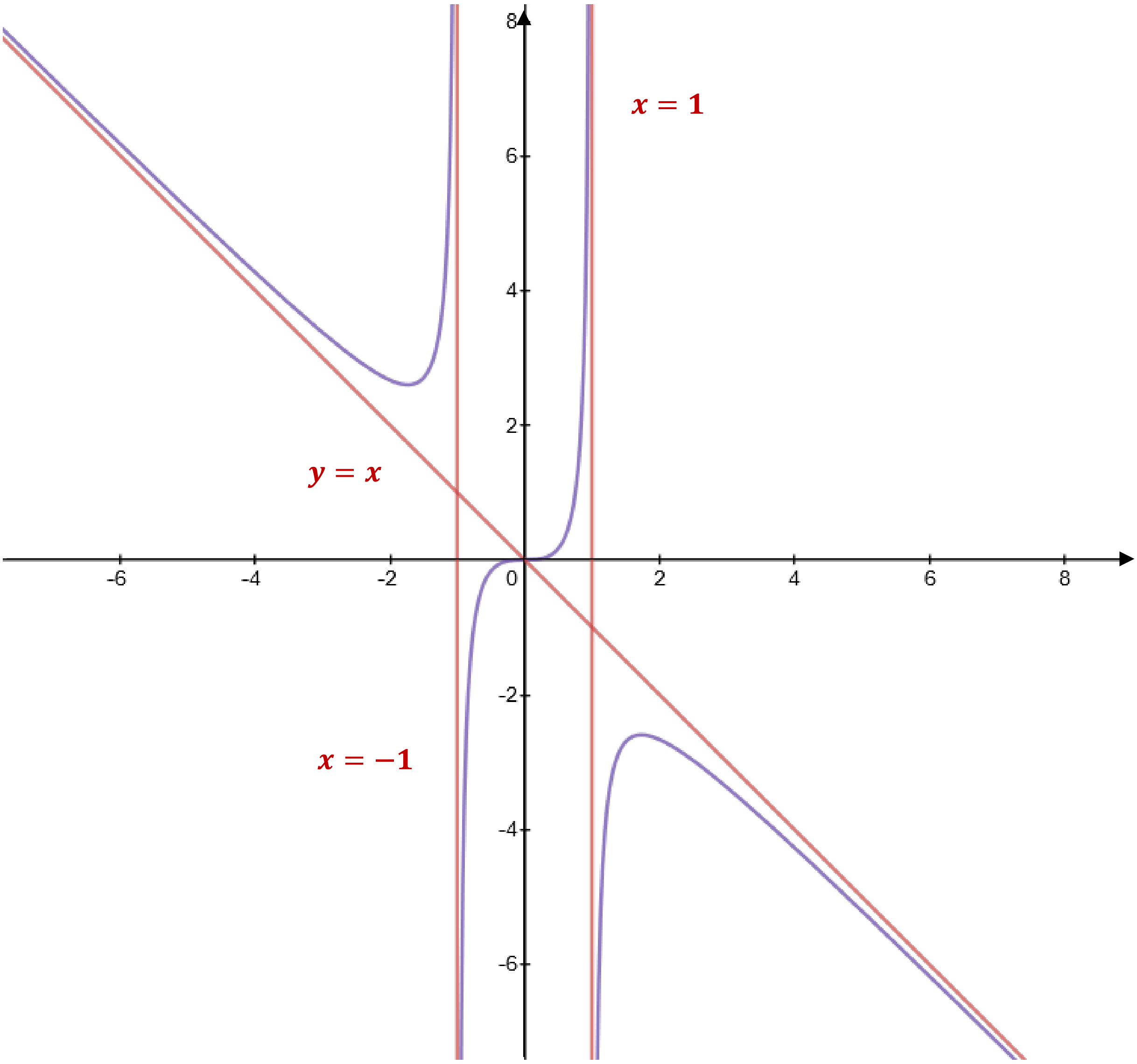
rising point of inflexion at $(0,0)$

x	$\rightarrow$	0	$\rightarrow$
$f''(x)$	-	0	+
Concavity	down		up

3. Vertical asymptotes at $x=\ \pm 1$
   Using algebraic long division $f(x)=\frac{x^3}{1-x^2}=-x+\frac{x}{1-x^2}$
   There is an oblique asymptote at $y=-x$

   As $x\rightarrow\infty^+,\ y\ \rightarrow -\infty+0^+$, graph approaches $y=-x$ from below.

   As $x\rightarrow\infty^-,\ y\ \rightarrow \infty+0^-$ graph approaches $y=-x$ from above.



Ideally this sketch should include the stationary points $(-\sqrt{3}, 2.598)$, and $(\sqrt{3}, -2.598)$.