Vectors

- Calculate the scalar and vector products for vectors in three dimensions.
- Know that $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

Revising work at higher level

- Any vector can either be denoted in component form $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ or using the unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ where $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} - u_3\boldsymbol{k}$

- The magnitude of vector $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is denoted by $|\boldsymbol{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

- A unit vector has a magnitude of 1. A unit vector in the same direction as vector $\boldsymbol{a}$ is denoted by $\boldsymbol{u_a}$

- The scalar or dot product and is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$, where

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \quad \text{or} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The angle θ between two vectors positioned tail to tail can be determined using

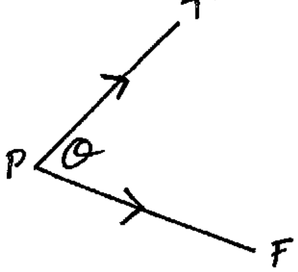
$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

For $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}| \cos 0 = |\boldsymbol{a}|^2$

If $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular

Example

Given the following points $T\ (-1,2,5)$, $P\ (0,5,-3)$ and $F\ (12,3,2)$, calculate the size of angle TPF.



$$\vec{PT} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix} \quad \vec{PF} = \begin{pmatrix} 12 \\ -2 \\ 5 \end{pmatrix}$$

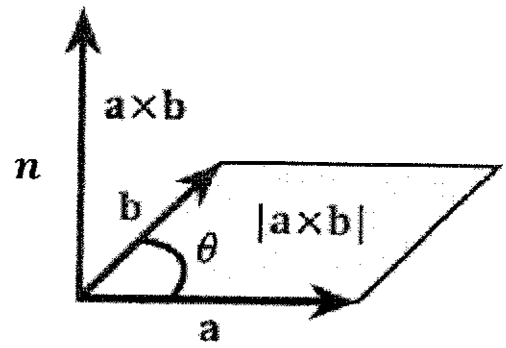
$$\cos(TPF) = \frac{34}{\sqrt{74}\sqrt{173}}$$

$$TPF = 72.5^\circ$$

p281 Ex 15.1

Learning to find the vector product

When two non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ define a plane and $\boldsymbol{n}$ is the unit vector perpendicular to this plane, then vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{n}$ form a right-handed system of vectors.



The **vector** or **cross product** occurs when $\boldsymbol{a}$ and $\boldsymbol{b}$ multiply and the resulting product is a vector with the same sense and direction as $\boldsymbol{n}$ and a magnitude $|\boldsymbol{a} \times \boldsymbol{b}|$ equal to the area of the parallelogram defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

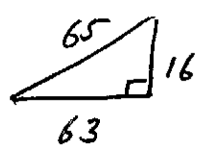
- The vector product is calculated using $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{n}|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$
- The magnitude of the vector product $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$ gives the area of a parallelogram defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
- Parallel vectors have a vector product of zero so, if $\boldsymbol{a} \neq 0, \boldsymbol{b} \neq 0$ **and** $|\boldsymbol{a} \times \boldsymbol{b}| = 0$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel
  $|\boldsymbol{a} \times \boldsymbol{a}| = 0$ (the area of the parallelogram is zero.)
- The unit vectors $\boldsymbol{i}, \boldsymbol{j}, and\ \boldsymbol{k}$ form a right-handed system of vectors where
  - $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}|\boldsymbol{i}||\boldsymbol{j}| \sin 90 = \boldsymbol{k}$, $\quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}|\boldsymbol{j}||\boldsymbol{k}| \sin 90 = \boldsymbol{i}$ etc

Example 1

Let R and T be the points (0,3,4) and (0,5,12), find the magnitude and direction of $\boldsymbol{r} \times \boldsymbol{t}$, **where r** is the position vector of R and **t** is the position vector of T.

$\underline{r} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \quad \underline{t} = \begin{pmatrix} 0 \\ 5 \\ 12 \end{pmatrix}$

$\cos\theta = \dfrac{\underline{r} \cdot \underline{t}}{|\underline{r}||\underline{t}|} = \dfrac{63}{65}$

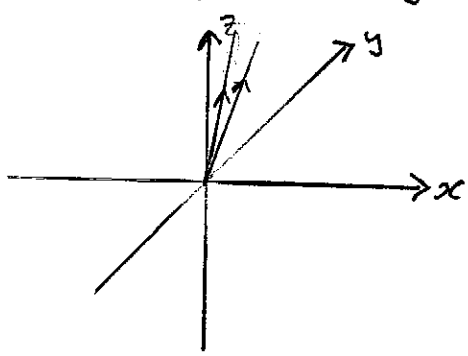


$|\underline{r}| = 5 \quad |\underline{t}| = 13$

$\sin\theta = \dfrac{16}{65}$

$|\underline{r} \times \underline{t}| = 5 \times 13 \times \dfrac{16}{65} = 16$

Direction: $\perp$r to y-z plane



Right-hand rule indicates +ve x direction.

$\underline{r} \times \underline{t} = 16\underline{i}$

p284 Ex 15.2 Q1

Vector Product in Component form

Given two vectors $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the vector product $\boldsymbol{a} \times \boldsymbol{b}$ can be found using the determinant of the 3×3 matrix $\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$, where the resulting vector is

$$\boldsymbol{a} \times \boldsymbol{b} = \mathbf{i} \times \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} - \boldsymbol{j} \times \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} + \boldsymbol{k} \times \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

Or $$\boldsymbol{i}(a_2b_3 - a_3b_2) - \boldsymbol{j}(a_1b_3 - a_3b_1) + \boldsymbol{k}\,(a_1b_2 - a_2b_1) = \begin{pmatrix} (a_2b_3 - a_3b_2) \\ (a_1b_3 - a_3b_1) \\ (a_1b_2 - a_2b_1) \end{pmatrix}$$

Check the answer to **example 1** above using the component definition of the vector product.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 3 & 4 \\ 0 & 5 & 12 \end{vmatrix} = \underline{i}\begin{vmatrix} 3 & 4 \\ 5 & 12 \end{vmatrix} - \underline{j}\begin{vmatrix} 0 & 4 \\ 0 & 12 \end{vmatrix} + \underline{k}\begin{vmatrix} 0 & 3 \\ 0 & 5 \end{vmatrix}$$

$$= (36 - 20)\underline{i} - 0\underline{j} + 0\underline{k}$$

$$= 16\,\underline{i}$$

Example 2 If $\boldsymbol{a} = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, find $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 7 & -1 \\ 2 & -2 & 1 \end{vmatrix} = \underline{i} \begin{vmatrix} 7 & -1 \\ -2 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 3 & 7 \\ 2 & -2 \end{vmatrix}$$

$$= (7-2)\underline{i} - (3+2)\underline{j} + (-6-14)\underline{k}$$

$$= 5\underline{i} - 5\underline{j} - 20\underline{k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -2 & 1 \\ 3 & 7 & -1 \end{vmatrix} = \underline{i} \begin{vmatrix} -2 & 1 \\ 7 & -1 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & -2 \\ 3 & 7 \end{vmatrix}$$

$$= (2-7)\underline{i} - (-2-3)\underline{j} + (14+6)\underline{k}$$

$$= -5\underline{i} + 5\underline{j} + 20\underline{k}$$

$$\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$$

The vector product is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ so the scalar product can be used to check your answer $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{b} = \boldsymbol{0}$

p286 Ex 15.3

Learning to find the scalar triple product

The scalar triple product is the scalar product of one vector with the vector product of two other vectors *i.e.* $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$.
This is **denoted by** $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$

This can be calculated from $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ or $\det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example 1 Find the scalar triple product for $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{c} = 2\boldsymbol{i} + 2\boldsymbol{j}$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \underline{k} \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot -4\underline{k}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix}$$

$$= 4$$

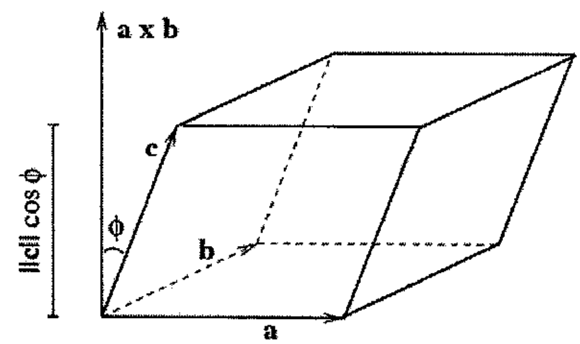
Or $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix}$

$$= -1 \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= -1(-2-2)$$

$$= 4$$

Volume of a parallelepiped. This is a 3D shaped bounded by the vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$



The volume of this shape is $V = ah$

where a is the area of the base of the parallelepiped $|\boldsymbol{a} \times \boldsymbol{b}|$

and h is the perpendicular distance between the planes $h = |\boldsymbol{c}| \cos\theta$

$$V = ah = |\boldsymbol{a} \times \boldsymbol{b}| \cdot |\boldsymbol{c}| \cos\theta = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$$

The volume of this parallelepiped is the **absolute value** of the scalar triple product $[\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}]$

Example 2 Find the volume of a parallelepiped bounded by the vectors

$$\boldsymbol{u} = \boldsymbol{i} + 6\boldsymbol{k}, \quad \boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, \quad \boldsymbol{w} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}.$$

$$\text{Volume} = |\underline{u} \cdot (\underline{v} \times \underline{w})|$$

$$= \begin{vmatrix} 1 & 0 & 6 \\ 2 & 1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} - 0 + 6 \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$$

$$= 1(1 + 4) + 6(-4 - 3)$$

$$= |5 - 42|$$

$$= |-37|$$

$$= 37 \text{ cubic units.}$$

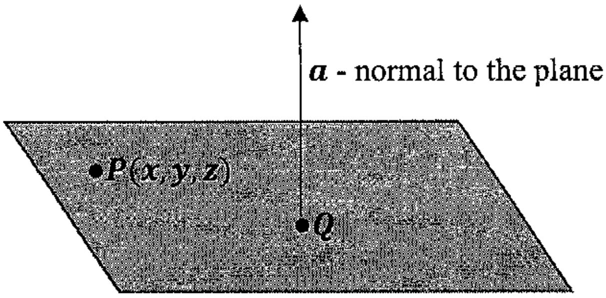
- The scalar triple product is commutative as $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$
- If any of $\boldsymbol{a}$, $\boldsymbol{b}$ or $\boldsymbol{c}$ are zero <u>**or**</u> if any two of $\boldsymbol{a}$, $\boldsymbol{b}$ or $\boldsymbol{c}$ are parallel then the scalar triple product is defined as **zero**

p288 Ex 15.4 Q 1-5

Learning to find the Equation of a plane in 3 dimensions

- Know the equation of a plane in Cartesian, vector and parametric form.
- Find the angle between two planes

Given any plane π where $P(x, y, z)$is a point on the plane and $\boldsymbol{a}$ is a normal to the plane passing through point Q



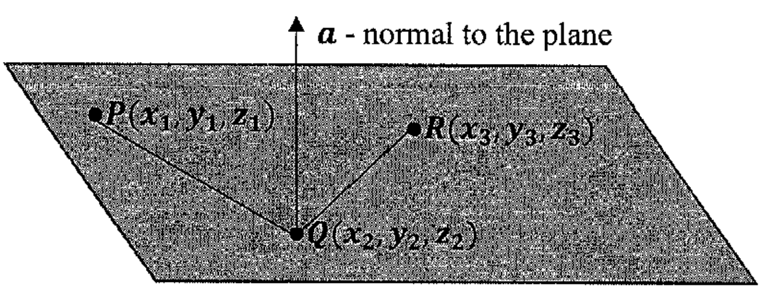
The cartesian equation of a plane is $ax + by + cz = d$, where $d = \boldsymbol{a} \cdot \boldsymbol{p}$

Example 1 Find the equation of the plane with normal $\boldsymbol{a} = 3\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k}$ and $(1,3,-1)$, a point lying on the plane.

$$\begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = 25$$

The equation is $3x + 7y - z = 25$

Given any 3 points P, Q and R on a plane , use these points to find two vectors and use the cross product to find the normal to the plane



Example 2

(a) Find the equation of the plane passing through points $A(2,1,-1), B(1,-1,2)$ and $C(0,-2,0)$.

(b) Determine if point $S(3,-2,3)$ lies on this plane.

(a) $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ -2 & -3 & 1 \end{vmatrix} = \underline{i}\begin{vmatrix} -2 & 3 \\ -3 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix}$

$= (-2+9)\underline{i} - (-1+6)\underline{j} + (3-4)\underline{k}$

$= 7\underline{i} - 5\underline{j} - \underline{k}$

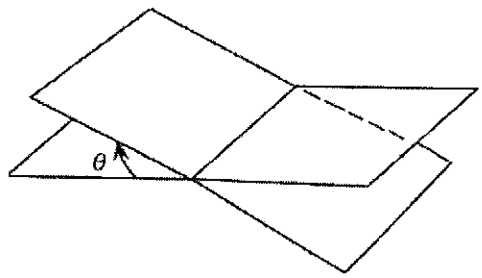
$\begin{pmatrix} 7 \\ -5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 10$ The equation is $7x - 5y - z = 10$

(b) $7(3) - 5(-2) + 3 = 21 + 10 - 3 \neq 10$ The point S does not lie on the line.

p291 Ex 15.5 Q1, 4, 5, 6

Angle between two planes

When two planes π_1 and π_2 meet, then their intersection is a straight line and the angle between the planes is θ the dihedral angle



The angle between the normal to these planes is **equal** to the dihedral angle. The angle between the planes is calculated using the scalar product

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Example 3

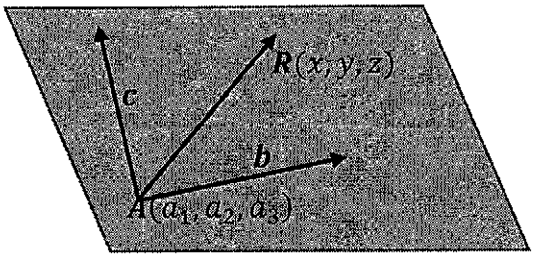
Find the angle between the planes $x + 2y - 3z = 7$ and $2x + 4y + 6z + 9 = 0$

$\underline{n}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ $\underline{n}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ $\cos\theta = \frac{-8}{\sqrt{14}\sqrt{56}}$

$|\underline{n}_1| = \sqrt{14}$ $|\underline{n}_2| = \sqrt{56}$ $\theta = 106.6°$

p293 Ex 15.6

Vector equation of a plane



For plane π_1 where A is a known point, R is any point on the plane $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-parallel vectors which lie in the plane.

Since <u>any</u> point in the plane can be reached from A via a multiple of **b** followed by a multiple of **a**, the vector equation of the planes takes the form

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + u\boldsymbol{c}, \qquad \boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + u\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

This can be rewritten to give the **parametric equation** of a plane

$$\boldsymbol{r} = (a_1 + tb_1 + uc_1)\boldsymbol{i} + (a_2 + tb_2 + uc_2)\boldsymbol{j} + (a_3 + tb_3 + uc_3)\boldsymbol{k}$$

Example 1

Find the equation of the plane through the points $A(1, -1, 3)$ $B(4, 1, -2)$ and $C(-1, -1, 1)$ in vector form.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + u\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$$

or $\underline{r} = (1 + 3t - 2u)\underline{i} + (-1 + 2t)\underline{j} + (3 - 5t - 2u)\underline{k}$

Example 2 Find the cartesian equation of the plane with the parametric equation

$$\boldsymbol{r} = (1 - 3t + 3u)\boldsymbol{i} + (2 + t + 2u)\boldsymbol{j} + (-1 + 3t + 3u)\boldsymbol{k}$$

$$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} + u\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & 1 & 3 \\ 3 & 2 & 3 \end{vmatrix}$$

$$= \underline{i}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} - \underline{j}\begin{vmatrix} -3 & 3 \\ 3 & 3 \end{vmatrix} + \underline{k}\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (3 - 6)\underline{i} - (-9 - 9)\underline{j} + (-6 - 3)\underline{k}$$

$$= -3\underline{i} + 18\underline{j} - 9\underline{k}$$

$$\begin{pmatrix} -3 \\ 18 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 42$$

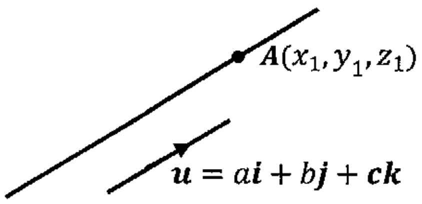
The equation is $-3x + 18y - 9z = 42$

or $-x + 6y - 3z = 14$

Learning to find the equations of a line

- Know the equation of a line in three dimensions in vector, parametric and symmetric form.
- Find the angle between two lines or a line and a plane.

The **vector equation of a straight line** (in 3 dimensions) is defined by a point on the line and the components of a vector parallel to the line



Point A has position vector $\boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

Vector u is parallel to the line $\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

The vector equation of this line is $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{u}$, $\boldsymbol{p} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

The parametric equation of this line is $x = x_1 + ta$

$y = y_1 + tb$

$z = z_1 + tc$

The symmetric or canonical equation of this line is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t$

Notes:

- *Sometimes t is omitted from the symmetric form.*
- *If one of a, b or c is zero, then use the vector equation or the parametric equation for this straight line.*

Example 1

A straight line passes through the point $(1,3,-5)$ and is parallel to vector $3\boldsymbol{i} + 5\boldsymbol{j} + 11\boldsymbol{k}$. Find the equation of this straight line and establish if point $(4,8,6)$ lies on this line.

$\underline{p} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 11 \end{pmatrix}$

For the point to lie on the line $\begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ 11 \end{pmatrix}$

i.e. $4 = 1 + 3t \Rightarrow t = 1$

$8 = 3 + 5t \Rightarrow t = 1$

$6 = -5 + 11t \Rightarrow t = 1$

The equations are consistent so the point lies on the line.

Example 2

A line passes through the points $A(1,1,0)$ and $B(-1,2,2)$. Find the equation of line AB

$$\vec{AB} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

The equation is $\underline{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

or $x = 1 - 2t$
$y = 1 + t$
$z = 2t$

or $\dfrac{x-1}{-2} = \dfrac{y-1}{1} = \dfrac{z}{2} \; (=t)$

p298 Ex 15.8	Q1, 3, 5
p298 Ex 15.9	Q2 and 5

Learning to find the intersection of a line and a plane

To find the point of intersection $\boldsymbol{P}$ between a line and a plane, substitute the parametric equation of the line $x = x_1 + ta,\ y = y_1 + tb,\ z = z_1 + tc$ into the equation of the plane and solve for t. This value for t can be substituted into the equation of the line to find point $\boldsymbol{P}$.

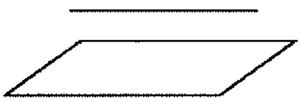
Example 1

Find the point of intersection between the plane $2x + y - 4z = 4$ and the line $\frac{x}{1} = \frac{y-2}{3} = \frac{z}{1}$.

$x = t \quad y = 3t + 2 \quad z = t$

$2x + y - 4z = 4$
$2t + (3t + 2) - 4t = 4$
$t + 2 = 4$
$t = 2$

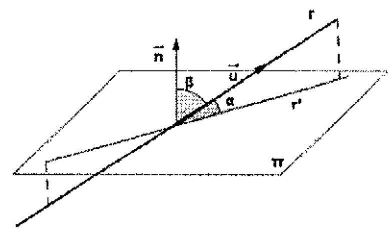
P.o.I $(2, 8, 2)$

Example 2 – find any points of intersection between the plane $2x + y - 4z = 4$ and the line $\frac{x-1}{1} = \frac{y-4}{2} = \frac{z}{1} = t$	**Example 3** – find any points of intersection between the plane $x + y + z = 3$ and the line $x = t + 3,\ y = 1 - 2t,\ z = t - 1$
parametric equation of the line $x = t + 1,\ y = 2t + 4,\ z = t$ substitution gives $2t + 2 + 2t + 4 - 4t = 4,\ 6 = 4$	Substitution gives $t + 3 - 2t + 1 + t - 1 = 3,\ 3 = 3$
Since there are **no** values of t which satisfy this equation. The line is parallel to the plane. 	Since **all** values of t satisfy this equation, this line is contained within the plane 

The acute angle between a line and a plane is the complement to the angle between the normal vector of the plane $\boldsymbol{n}$ and the direction vector of the line $\boldsymbol{u}$

Since $\cos(90° - \theta) = \sin\theta$

this angle is can also be found using $\sin\theta = \frac{|\boldsymbol{n} \cdot \boldsymbol{u}|}{|\boldsymbol{n}||\boldsymbol{u}|}$



Example 4

Determine the angle between the plane $x + y = 1$ and the line $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z}{2}$

$\underline{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\cos\theta = \frac{3}{\sqrt{2}\sqrt{9}} = \frac{1}{\sqrt{2}}$ $\theta = 45°$

The angle between the normal and the line is $45°$ so the angle between the plane and the line is $90 - 45 = 45°$

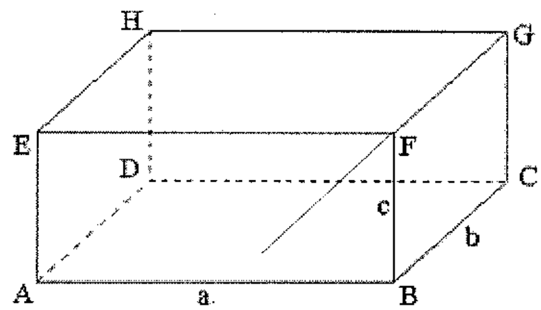
<u>OR</u> $\sin\theta = \frac{1}{\sqrt{2}}$ $\theta = 45°$

The angle between the plane and the line is $45°$.

p300 Ex 15.10 Q1, 2, 4 selected parts Q5

Learning to find the intersection of two lines

Two lines in 3 dimensional space can either be parallel, intersect at a point or be skew (not parallel, but will never intersect).



AB is parallel to EF, AB intersects BC at point B, AB and FG are skew.

To find a point of intersection:

1. Express the equations of both lines in parametric form using the parameters t_1 and t_2
2. Equate corresponding expressions for x, y and z producing three equations with two unknowns
3. Use two of the equations to find values for t_1 and t_2
4. Substitute these values into the third equation. If these values satisfy the equation this is the point of intersection. If they do not satisfy the equation then the lines do not intersect

Example

Show that the lines $x - 5 = -(y + 2) = z$ and $\frac{x-12}{5} = \frac{y+3}{-2} = \frac{z-5}{4}$ intersect and find the point of intersection.

L_1 $x = t_1 + 5$
$y = -t_1 - 2$
$z = t_1$

L_2 $x = 5t_2 + 12$
$y = -2t_2 - 3$
$z = 4t_2 + 5$

$t_1 + 5 = 5t_2 + 12$ ①
$-t_1 - 2 = -2t_2 - 3$ ②
① + ②
$3 = 3t_2 + 9$
$t_2 = -2$
$t_1 + 5 = -10 + 12$
$t_1 = -3$

verify in z coordinate
$z = t_1 = -3$
$z = 4t_2 + 5 = 4 \times -2 + 5 = -3$
Equation 3 is consistent so the lines intersect.

PoI $(2, 1, -3)$

Find also the angle made by these intersecting lines.

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$$

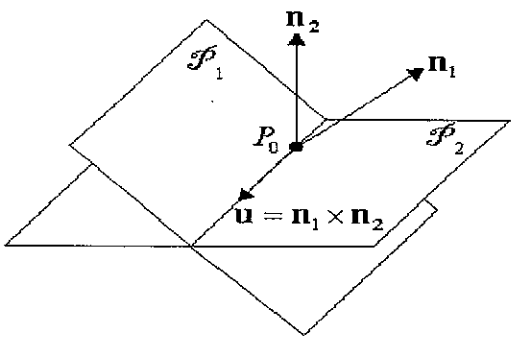
$$\cos\theta = \frac{11}{\sqrt{3}\sqrt{45}}$$

$$\theta = 18 \cdot 8^\circ$$

p302 Ex 15.11 Q 1-3

Learning to find the intersection of two planes

Two planes are either parallel or intersect in a single straight line L. To establish the equation of this line L we need a direction vector $\boldsymbol{u}$ and a point P on the line.



Since the direction vector $\boldsymbol{u}$ lies within both planes it is perpendicular to both normal. Thus, it can be found using the cross product of the normal vectors for both planes $\boldsymbol{u} = \boldsymbol{n_1} \times \boldsymbol{n_2}$

Line L will either cross the (x, y) plane (where $z = 0$) or be parallel to it so we can identify a point P by setting $z = 0$ in the equations of both planes and solving simultaneously to find $P = (x_1, y_{1,} 0)$

The equation of line L can then be represented in vector, parametric or symmetric form

Example

Find the equation of the line which represents the intersection of the planes
$\pi_{1:}\, 2x - y + z = 5$ and $\pi_2: x + y - z = 1$

$$\underline{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \underline{n}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{u} = \underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \underline{i} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \underline{j} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \underline{k} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= (1-1)\underline{i} - (-2-1)\underline{j} + (2+1)\underline{k}$$

$$= 3\underline{j} + 3\underline{k}$$

Point on the line: let $z = 0$

$$2x - y = 5$$
$$x + y = 1$$
$$3x = 6$$
$$x = 2$$
$$y = -1$$

$(2, -1, 0)$

p303 Ex 15.12 Q 1 and 3

The equation of the line is

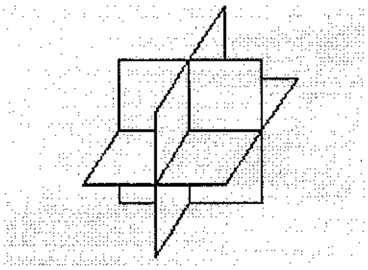
$$\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

Learning to find the intersection of 3 planes

To find the intersection of three planes we solve the equations of the planes simultaneously *i.e.* solve a system of three equations in three unknown. Use the planar equations to form an augmented matrix and proceed with Gaussian Elimination to solve for x, y and z

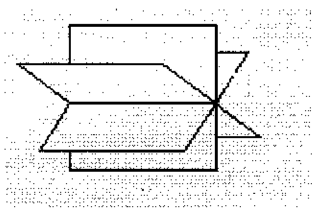
There are six different cases for the intersection three planes

When the intersection is a single point



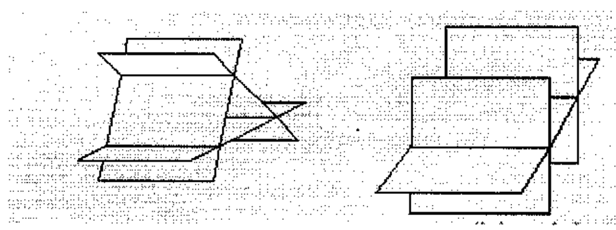
Unique solution shows that all three planes intersect at point $P(a, b, c)$

When the intersection is a line



Third equation id redundant (indicated by a row of zeros). If this is, the case let $z = t$ and use the first two rows to find parametric equation for the line of intersection.

When the intersection is three or two lines

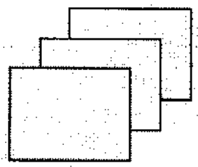


The third equation shows inconsistency and the system has no solutions.
If this happens, let $z = t$ and examine the planes in pairs (1&2, 1&3, 2&3) to find either three lines of intersections or two lines of intersection (where two planes are parallel).

When the intersection is a plane

Where the second and third rows in the matrix are all zeros. This leaves the top row of the matrix to give the equation of the plane of intersection

When there is no intersection



If there is more than one inconsistent set of equations then there is no solution. This usually occurs when you have a set of 3 parallel planes

p307 Ex 15.13 Q 1 and 2