X204/13/01

NATIONAL QUALIFICATIONS 2013

TUESDAY, 14 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mechanics 1 and 2)

Marks

Answer all the questions

Candidates should observe that g m s^{-2} denotes the magnitude of the acceleration due to gravity.

Where appropriate, take its magnitude to be 9·8 m s^{-2}.

A1. A particle is moving in a plane such that t seconds after the start of its motion, the velocity is given by $(3t\boldsymbol{i} + 5t^2\boldsymbol{j})$ m s^{-1}.

The particle is initially at the point $\left(\frac{1}{2}\boldsymbol{i} - 7\boldsymbol{j}\right)$ metres relative to a fixed origin O.

Find the distance of the particle from O when $t = 3$. **3**

A2. A ball of mass 0·5 kg is released from rest at a height of 10 metres above the ground.

If the ball reaches 2·5 metres after its first bounce, calculate the size of the impulse exerted by the ground on the ball. **4**

A3. A particle of mass 3 kilograms moves under the action of its own weight and a constant force $\mathbf{F} = (3\boldsymbol{i} + 5{\cdot}4j)$ where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the horizontal and vertical directions respectively.

Initially the particle has velocity $(2\boldsymbol{i} - \boldsymbol{j})$ m s^{-1} as it passes through a point A. The particle passes through B after 4 seconds. Find the work done to move the particle from A to B. **5**

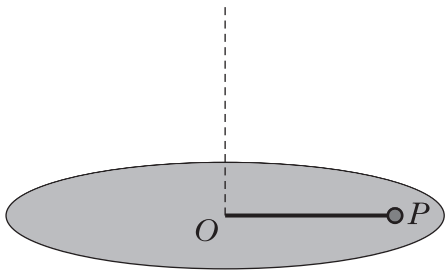
A4. A go-kart of mass 100 kilograms accelerates at 3 m s^{-2} at the instant when its speed is 5 m s^{-1} and the engine's power is at a maximum.

Given that there is a total resistance to motion of 60 newtons throughout the go-kart's motion, find the maximum speed which the go-kart can achieve. **4**

A5. A piano of mass 160 kilograms is resting on a rough plane inclined at an angle $\theta°$ to the horizontal, where $\tan\theta° = \frac{7}{24}$. When a removal man applies a horizontal force of 850 newtons, the piano is just on the point of moving up the plane. Find the value of the coefficient of friction between the piano and the surface of the plane. **4**

When the removal man increases the horizontal force to 1000 newtons, the piano begins to accelerate up the plane, along the line of greatest slope. How far does the piano travel in 3 seconds? **2**

A6. A rough disc rotates in a horizontal plane with a constant angular velocity ω about a fixed vertical axis through the centre O. A particle of mass m kilograms lies at a point P on the disc and is attached to the axis by a light elastic string OP of natural length a metres and modulus of elasticity $2mg$.



The particle is at a distance of $\frac{5a}{4}$ from the axis and the coefficient of friction between P and the disc is $\frac{3}{20}$.

Find the range of values for ω such that the particle remains stationary on the disc. **5**

A7. A light elastic string of natural length l metres hangs from a fixed point O with a particle of mass m kilograms attached at its lower end. In equilibrium the string is extended by e metres.

The particle is then pulled down a further distance a metres where $a < e$ and released.

Show that the ensuing motion is simple harmonic and state the period of the motion. **4**

The maximum velocity of the particle during motion is $\frac{1}{2}\sqrt{ge}$. Find an expression for the amplitude of the motion in terms of e. **2**

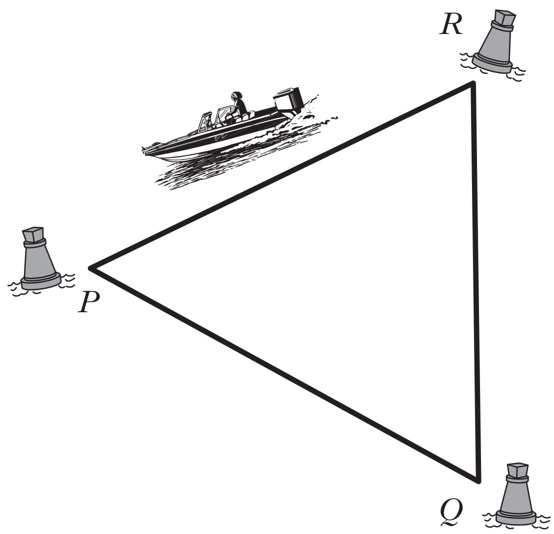
[Turn over

Marks

A8. A smooth solid hemisphere of radius a metres is fixed with its plane face on a horizontal table and its curved surface uppermost. A particle P of mass m kilograms is placed at the highest point on the hemisphere and given an initial horizontal speed $\sqrt{\frac{ag}{2}}\ \mathrm{m\,s^{-1}}$. The particle moves along the curved surface of the hemisphere until it leaves the surface at Q.

Calculate the angle between the tangent at Q and the horizontal, and find an expression for the speed of the particle at Q. **6**

A9. As part of a race, a speedboat has to round three buoys P, Q and R, starting at P and travelling anti-clockwise. The buoys are 200 metres from each other with R due North of Q and P lying to the west of the line QR. In still water, the speedboat travels at $20\,\text{m}\,\text{s}^{-1}$. The water current is steady at $5\,\text{m}\,\text{s}^{-1}$ flowing from due West.



Find the mean speed for one complete lap of the course. **9**

A10. Two projectiles are launched simultaneously from points A and B, where B is due East of A and situated on the same horizontal plane as A. The projectile launched from point A is projected towards B with speed $90\,\text{m}\,\text{s}^{-1}$ at an angle of $30°$ to the horizontal. The projectile from point B is projected towards A with speed $50\,\text{m}\,\text{s}^{-1}$ at an angle $\theta°$ to the horizontal.

The two projectiles collide in mid-air at a distance d metres horizontally from point A.

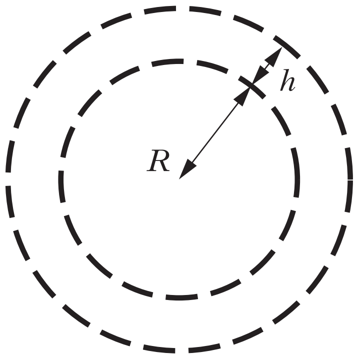
Show that the height h at this point of collision is $h = \dfrac{d\left(4050\sqrt{3} - gd\right)}{12150}$. **4**

Find the angle of projection, $\theta°$, at which the projectile from B is launched. **4**

The projectiles collide 5 seconds after launch. Calculate the distance between A and B. **2**

[Turn over

A11. A body of fixed mass m kilograms is projected vertically upwards from a point on the surface of a planet with an initial speed of $u\,\mathrm{m\,s^{-1}}$.



Assuming that the gravitational force on the body is $\dfrac{GMm}{d^2}$ where d metres is the distance from the centre of the planet, show that the speed of the body when it has reached a height h metres above the surface is given by $v = \sqrt{u^2 - \dfrac{2GMh}{R(R+h)}}$, where M kilograms is the mass of the planet, R metres is the radius of the planet and G is the gravitational constant. **5**

Find an expression for the maximum height, H, reached by the body. **3**

Show that the escape speed necessary for the body to continue into space can be written in the form $u = k\sqrt{\dfrac{GM}{R}}$ and state the value of k. **2**

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions

B1. Given that $y = \sin(e^{5x})$, find $\dfrac{dy}{dx}$. **2**

B2. Matrices are given as

$$A = \begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} y & 3 \\ -1 & 2 \end{pmatrix}.$$

(*a*) Write $A^2 - 3B$ as a single matrix. **2**

(*b*) (i) Given that C is non-singular, find C^{-1}, the inverse of C. **2**

(ii) For what value of y would matrix C be singular? **1**

B3. Use integration by parts to obtain

$$\int \frac{\ln x}{x^3}\, dx$$

where $x > 0$. **4**

B4. (*a*) State $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ in terms of n.

Hence show that

$$\sum_{r=1}^{n}\left(r^3 - 3r\right) = \frac{n(n+1)(n-2)(n+3)}{4}.$$ **4**

(*b*) Use the above result to evaluate $\sum_{r=5}^{15}\left(r^3 - 3r\right)$. **2**

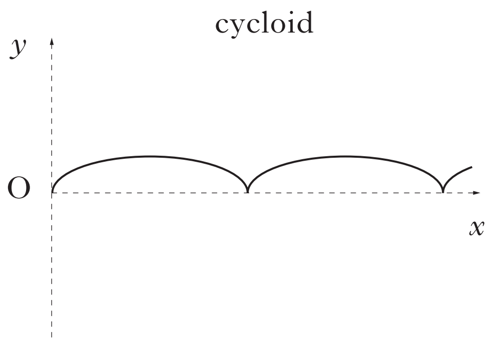
Marks

B5. Find the general solution of the differential equation

$$\frac{1}{x}\frac{dy}{dx} + 2y = 6,\ x \neq 0.$$ **6**

B6. The cycloid curve below is defined by the parametric equations

$$x = t - \sin t,\ y = 1 - \cos t.$$



(*a*) Find $\frac{dy}{dx}$ in terms of t. **2**

(*b*) Show that the value of $\frac{d^2y}{dx^2}$ is always negative, in the case where $0 < t < 2\pi$. **5**

(*c*) A particle follows the path of the cycloid where t is the time elapsed since the particle's motion commenced.

Calculate the speed of the particle when $t = \frac{\pi}{3}$. **2**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]