X204/701

NATIONAL
QUALIFICATIONS
2005

MONDAY, 23 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

    Section A assesses the Units Mechanics 1 and 2
    Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

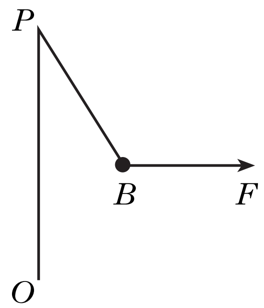
Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

Candidates should observe that g m s^{-2} denotes the magnitude of the acceleration due to gravity.
Where appropriate, take its magnitude to be 9·8 m s^{-2}.

A1. A ball B of weight 9 newtons is attached to one end of a light inextensible string. The other end of the string is attached to P, the top of a fixed vertical pole OP.



By exerting a horizontal force of magnitude F newtons, the ball is held in equilibrium, with the string taut and $\angle OPB = 30°$.

Calculate:

(*a*) the tension in the string; **2**

(*b*) the value of F. **2**

A2. A ball is projected vertically from ground level. The ball attains a maximum height of 49 metres before returning to the ground.

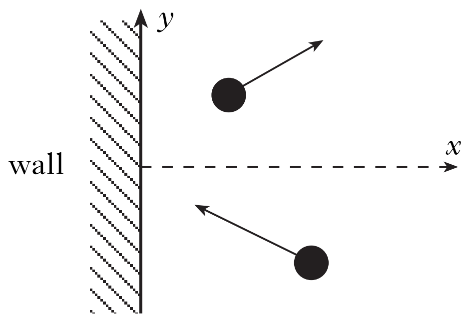
Assuming only the action of gravity, calculate the time of flight of the ball. **5**

A3. A particle executes simple harmonic motion about a point O. The magnitude of the maximum acceleration is 1 m s^{-2} and the maximum speed is 4 m s^{-1}.

Calculate the period of the motion. **4**

Marks

A4. A ball of mass 0·01 kg collides with a fixed vertical wall. Immediately before the collision the velocity of the ball is $-3\mathbf{i} + 4\mathbf{j}$, and just after the collision the velocity is $2\mathbf{i} + 3\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions of the rectangular coordinate system shown below and speeds are measured in $\mathrm{m\,s^{-1}}$.



Calculate the magnitude of the impulse exerted on the ball by the wall. **3**

A5. The velocity of an ice skater relative to a rectangular coordinate system with origin O, is given by

$$\mathbf{v} = 3(t^2 - 4t + 2)\mathbf{i} + 4\mathbf{j},$$

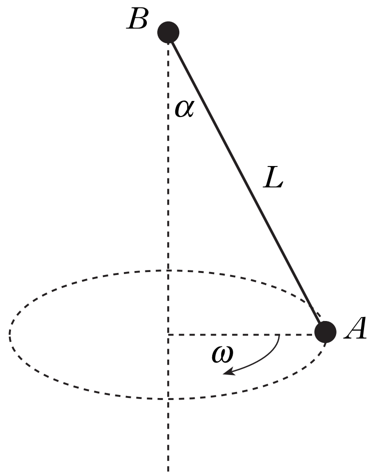
where $\mathbf{i}$, $\mathbf{j}$ are unit vectors in the Ox and Oy directions, t seconds is the time and the speed is measured in $\mathrm{m\,s^{-1}}$. Initially the skater has position vector $-4\mathbf{j}$.

(*a*) Find the time at which the acceleration is instantaneously equal to zero. **2**

(*b*) Calculate the distance of the skater from O when the acceleration is instantaneously equal to zero. **4**

[Turn over

A6. A ball of mass m kg is attached to one end A of a light inextensible string of length L metres. The other end of the string is attached to a fixed point B. The ball moves, with string taut, in a horizontal circle with constant angular speed ω radians per second as shown in the diagram. During this motion, the string is inclined at an angle α to the downward vertical through B where $\tan\alpha = \frac{5}{12}$.



(a) Find the tension in the string in terms of m and g. **2**

(b) Find an expression for ω in terms of g and L. **3**

A7. A particle of mass 2 kg is accelerated horizontally from rest at a point O by a force $8t\mathbf{i}$, whose magnitude is measured in newtons and where $\mathbf{i}$ is the unit vector in the direction of motion and t seconds is the time from the start of the motion.

(a) Find the velocity, $\mathbf{v}$, of the particle as a function of time t. **2**

(b) Calculate the work done on the particle in the first second of the motion. **3**

A8. A bungee jumper of mass m kg falls vertically from rest from a high bridge. One end of an elastic rope is attached to the jumper, the other end to the bridge at the point where the jumper commences her fall. The natural length of the rope is l metres and the modulus of elasticity is $12mg$ newtons.

At the moment when the jumper is brought instantaneously to rest by the rope, the extension of the rope is a metres.

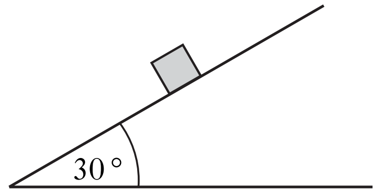
(a) Neglecting the effect of air resistance, use conservation of energy to show that the extension satisfies

$$6a^2 - la - l^2 = 0.$$ **3**

(b) Hence find, in terms of l, the distance the jumper falls before first coming instantaneously to rest. **3**

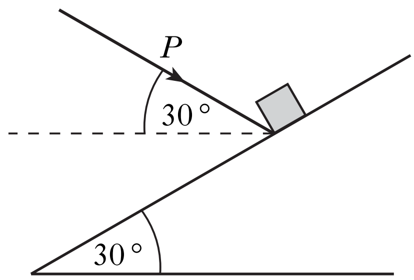
Marks

A9. (*a*) A box of mass m kg is placed on a rough plane inclined at $30°$ to the horizontal. The coefficient of friction between the box and the plane is μ.



Given that the box remains in equilibrium, show that $\mu \geq \dfrac{1}{\sqrt{3}}$. **3**

(*b*) The same box is kept in equilibrium on another rough plane, which is also inclined at $30°$ to the horizontal, by the action of a force of magnitude P newtons as shown in the diagram below. This force is acting up the plane at an angle of $30°$ to the horizontal. The coefficient of friction between the box and this plane is 0·5 and the box is on the point of slipping down the plane.



(i) Show that the reaction force normal to the inclined plane has magnitude given by

$$R = \frac{\sqrt{3}}{2}(mg + P) \text{ newtons.}$$ **2**

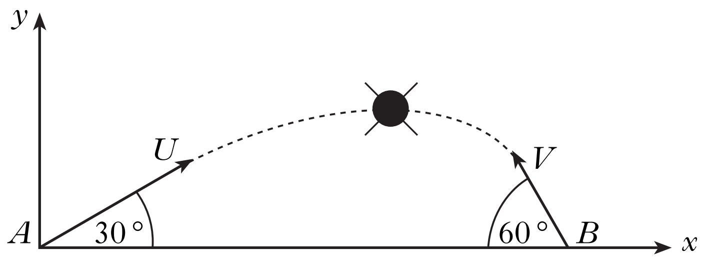
(ii) Show further that

$$P = \frac{(2-\sqrt{3})mg}{2+\sqrt{3}} \text{ newtons.}$$ **5**

[Turn over

Marks

A10. Two points A and B are a distance L metres apart on horizontal ground. A ball is thrown from A towards B with speed $U\ \mathrm{m\,s^{-1}}$ at an angle of projection of $30°$. Simultaneously, a second ball is thrown from B towards A with speed $V\ \mathrm{m\,s^{-1}}$ and angle of projection $60°$.



(*a*) Using the coordinate system shown in the diagram:

(i) write down expressions in terms of U and t for the x and y coordinates of the ball thrown from A at time t seconds after projection; **2**

(ii) show that at time t, the x-coordinate of the ball thrown from B is $x = L - \frac{1}{2}Vt$ and write down the corresponding expression for the y-coordinate. **2**

(*b*) The balls collide before reaching the ground.

(i) Show that $U = \sqrt{3}V$. **2**

(ii) Find an expression for the horizontal distance from A at which the collision takes place, giving your answer in terms of L. **4**

A11. A particle is projected horizontally from the origin, O, along the positive x-axis with initial speed $1\ \mathrm{m\,s^{-1}}$. The particle has acceleration $4(4x - 1)\mathbf{i}\ \mathrm{m\,s^{-2}}$, where x metres ($0 \leq x \leq \frac{1}{4}$) is the distance of the particle from O after time t seconds and $\mathbf{i}$ is a unit vector in the direction of the x-axis.

(*a*) Show that the speed, $v\ \mathrm{m\,s^{-1}}$, of the particle is given by

$$v = 1 - 4x.$$ **5**

(*b*) Hence show that

$$x = \frac{1}{4}(1 - e^{-4t}).$$ **5**

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Differentiate, and simplify as appropriate,

(*a*) $f(x) = \exp(\tan \frac{1}{2}x)$, where $-\pi < x < \pi$, **3**

(*b*) $g(x) = (x^3 + 1) \ln (x^3 + 1)$, where $x > 0$. **3**

B2. Given that $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$, show that $A^2 - A = kI$ for a suitable value of k, where I is the 2×2 unit matrix. **3**

B3. A curve is defined by the parametric equations $x = 5t^2 - 5$, $y = 3t^3$.

Find the value of t corresponding to the point $(0, -3)$ and calculate the gradient of the curve at this point. **2, 3**

B4. Expand and simplify $\left(2a - \frac{3}{a}\right)^4$. **3**

B5. Express $\dfrac{x^2 + 3}{x(1 + x^2)}$ in partial fractions. **3**

Hence obtain $\displaystyle\int_{1/2}^{1} \frac{x^2 + 3}{x(1 + x^2)}\, dx$. **3**

B6. (*a*) Given the differential equation

$$\sin x \frac{dy}{dx} - 2y \cos x = 0,$$

find the general solution, expressing y explicitly in terms of x. **4**

(*b*) Find the general solution of

$$\sin x \frac{dy}{dx} - 2y \cos x = 3 \sin^3 x.$$ **5**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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