SCOTTISH
QUALIFICATIONS
AUTHORITY

2005 Applied Mathematics

Advanced Higher – Mechanics

Finalised Marking Instructions

These Marking Instructions have been prepared by Examination Teams for use by SQA Appointed Markers when marking External Course Assessments.

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 1, 1M, 1, 1 means a method mark for the product rule (and then a mark for each of the terms). E is shorthand for error. In question 3, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Section A – Mechanics

A1. (a) Resolving vertically: $T\cos 30° = 9 \Rightarrow T = \frac{18}{\sqrt{3}} = 6\sqrt{3} \approx 10.4\text{N}$. **1,1**

(b) Resolving horizontally: $F = T\cos 60° \Rightarrow F = \frac{1}{2}6\sqrt{3} = 3\sqrt{3} \approx 5.2\,\text{N}$. **1,1**

A2. Let the initial velocity be V m s^{-1}.
Applying $v^2 = u^2 + 2as$ gives $0^2 = V^2 - 2gh \Rightarrow V = \sqrt{98g}$. **1,1**
Now apply $s = ut + \frac{1}{2}at^2$. This gives $0 = T\left(V - \frac{1}{2}gT\right)$. Thus **M1**

$$T = \frac{2V}{g} = 2\sqrt{\frac{98}{g}} \approx 2\sqrt{10}$$

so the total time of flight is $2\sqrt{10} \approx 6.3$ seconds. **2E1**

A3. $a_{\text{max}} = \omega^2 a \Rightarrow \omega^2 a = 1$ **1**
$v_{\text{max}} = \omega a \Rightarrow \omega a = 4$ **1**
Dividing gives $\omega = \frac{1}{4}$ **1**
The period is $\frac{2\pi}{\omega} = 8\pi$. **1**

A4.

$$\mathbf{I} = m(\mathbf{v} - \mathbf{u})$$ **1**

$$= 0.01\,(2\mathbf{i} + 3\mathbf{j}) - (-3\mathbf{i} + 4\mathbf{j}))$$

$$= 0.01\,(5\mathbf{i} - \mathbf{j})$$ **1**

Thus

$$|\mathbf{I}| = 0.01\sqrt{5^2 + (-1)^2} \approx 0.051 \text{ kg m s}^{-1}$$ **1**

A5. (a)

$$\frac{d\mathbf{v}}{dt} = 3\,(2t - 4)\,\mathbf{i}$$ **1**

$$\text{When } \frac{d\mathbf{v}}{dt} = 0,\ t = 2.$$ **1**

(b)

$$\mathbf{r} = \int \mathbf{v}\,dt$$

$$= \int 3\left(t^2 - 4t + 2\right)\mathbf{i} + 4\mathbf{j}\,dt$$

$$= \left(t^3 - 6t^2 + 6t\right)\mathbf{i} + 4t\mathbf{j} + \mathbf{c}$$ **1**

When $t = 0$, $\mathbf{r} = -4\mathbf{j}$ so $\mathbf{c} = -4\mathbf{j}$ and **1**

$$\mathbf{r} = \left(t^3 - 6t^2 + 6t\right)\mathbf{i} + (4t - 4)\,\mathbf{j}$$

When $t = 2$

$$\mathbf{r} = -4\mathbf{i} + 4\mathbf{j}$$ **1**

Thus the distance from the skater to O is $|\mathbf{r}\,(2)| = 4\sqrt{2}$ metres. **1**

A6. (a) Resolving vertically: $T\cos\alpha = mg$ **1**

$$\Rightarrow T = \frac{mg}{\cos a} = \frac{13}{12}mg$$ **1**

(b) Central acceleration = $r\omega^2$, with

$$r = L\sin\alpha.$$ **1**

$$\text{so } T\sin\alpha = mr\omega^2$$ **1**

$$\omega^2 = \frac{T\sin\alpha}{mL\sin\alpha} = \frac{13g}{12L}$$ **1**

$$\Rightarrow \omega = \sqrt{\frac{13g}{12L}}.$$

A7. (a)

$$\mathbf{F} = m\mathbf{a}$$

$$\Rightarrow \mathbf{a} = 4t\mathbf{i}$$ **1**

$$\Rightarrow \mathbf{v} = 2t^2\mathbf{i} + \mathbf{c} \qquad (\text{but } \mathbf{c} = \mathbf{0})$$

$$\Rightarrow \mathbf{v} = 2t^2\mathbf{i}$$ **1**

(b)

$$\text{Work done} = \int \mathbf{F}.\mathbf{v}\,dt$$ **1**

$$= \int 16t^3 dt$$ **1**

$$= \left[4t^4\right]_0^1 = 4 \text{ N m}$$ **1**

Or:

$$\text{Work done} = \text{change in KE}$$ **1**

$$= \tfrac{1}{2} \times 2 \times 2^2 - 0$$ **1**

$$= 4 \text{ N m}$$ **1**

A8. (a) Taking the lowest point as the base line. Then at the bridge

Gravitational PE = $mg(l + a)$

Elastic PE = 0

KE = 0 **1**

At lowest point

Gravitational PE = 0

Elastic PE = $\dfrac{\lambda a^2}{2l} = \dfrac{12mga^2}{2l}$

KE = 0 **1**

So

$$mg(l + a) = \frac{6mga^2}{l}$$ **1**

$$\Rightarrow 6a^2 - la - l^2 = 0$$

(b)

$$6a^2 - la - l^2 = 0$$

$$(3a + l)(2a - l) = 0$$ **1**

As $a > 0$, $a = \frac{1}{2}l$ **1**

so the total 'drop' is $\frac{3}{2}l$. **1**

A9. (a) Resolving perpendicular to the plane:

$$R = mg\cos 30° = \frac{\sqrt{3}}{2}mg \qquad \mathbf{1}$$

Resolving parallel to the plane:

$$F = mg\sin 30° = \frac{1}{2}mg \qquad \mathbf{1}$$

Since, $F \leqslant \mu R$

$$\tfrac{1}{2}mg \leqslant \mu\tfrac{\sqrt{3}}{2}mg \qquad \mathbf{1}$$

$$\text{i.e. } \mu \geqslant \frac{1}{\sqrt{3}}.$$

(b) (i) Resolving perpendicular to the plane:

$$R = mg\cos 30° + P\cos 30° \qquad \mathbf{1,1}$$

$$= (mg + P)\frac{\sqrt{3}}{2}$$

(ii) Resolving parallel to the plane:

$$F + P\cos 60° = mg\sin 30° \qquad \mathbf{1,1}$$

$$F = \tfrac{1}{2}(mg - P)$$

But as friction is limiting, $F = \mu R$

$$\frac{1}{2}(mg - P) = \frac{1}{2}\left((mg + P)\frac{\sqrt{3}}{2}\right) \qquad \mathbf{1,1}$$

$$2(mg - P) = \sqrt{3}(mg + P)$$

$$\left(2 + \sqrt{3}\right)P = \left(2 - \sqrt{3}\right)mg \qquad \mathbf{1}$$

$$P = \frac{\left(2 - \sqrt{3}\right)mg}{\left(2 + \sqrt{3}\right)}$$

A10. (a) (i)

$$x = U\cos 30° t \qquad \mathbf{1}$$

$$= \tfrac{\sqrt{3}}{2}Ut$$

$$y = U\sin 30° t - \tfrac{1}{2}gt^2 \qquad \mathbf{1}$$

$$= \tfrac{1}{2}Ut - \tfrac{1}{2}gt^2$$

(ii) Horizontal distance from B is $V\cos 60° t = \tfrac{1}{2}Vt$.

So the x-coordinate is $L - \tfrac{1}{2}Vt$. **1**

The y-coordinate is $V\sin 60° t - \tfrac{1}{2}gt^2 = \tfrac{\sqrt{3}}{2}Vt - \tfrac{1}{2}gt^2$. **1**

(b) (i) At the time of collision, the heights are equal

$$\frac{1}{2}Ut - \frac{1}{2}gt^2 = \frac{\sqrt{3}}{2}Vt - \frac{1}{2}gt^2 \qquad \mathbf{1}$$

$$\frac{1}{2}Ut = \frac{\sqrt{3}}{2}Vt \qquad \mathbf{1}$$

$$U = V\sqrt{3}.$$

(ii) Collision when the x-coordinates are equal

$$L - \frac{1}{2}Vt = \frac{Ut\sqrt{3}}{2} \qquad \mathbf{1}$$

$$L - \frac{Ut}{2\sqrt{3}} = \frac{Ut\sqrt{3}}{2} \qquad \mathbf{1}$$

$$3Ut + Ut = 2\sqrt{3}L$$

$$Ut = \frac{\sqrt{3}L}{2} \qquad \mathbf{1}$$

So the distance from A is $\frac{3}{4}L$. **1**

A11. (a)

$$a = 4(4x - 1)$$

$$v\frac{dv}{dx} = 4(4x - 1) \qquad \mathbf{1}$$

$$\int v\,dv = 4\int(4x - 1)\,dx$$

$$\tfrac{1}{2}v^2 = 4(2x^2 - x) + c \qquad \mathbf{1}$$

When $x = 0, v = 1$, so $c = \frac{1}{2}$ and **1**

$$v^2 = 16x^2 - 8x + 1$$

$$= (1 - 4x)^2. \qquad \mathbf{1}$$

i.e. $v = 1 - 4x$, since $v > 0$. **1**

(b)

$$\frac{dx}{dt} = 1 - 4x \qquad \mathbf{1}$$

$$\int \frac{dx}{1 - 4x} = \int dt \qquad \mathbf{1}$$

$$-\frac{1}{4}\ln|1 - 4x| = t + c'$$

When $t = 0, x = 0$ so $c' = 0$. **1**

$$\ln|1 - 4x| = -4t$$

$$|1 - 4x| = e^{-4t} \qquad \mathbf{1}$$

$$4x = 1 - e^{-4t} \text{ since } 4x < 1 \qquad \mathbf{1}$$

$$x = \frac{1 - e^{-4t}}{4}.$$

Section B – Mathematics

B1. (a)

$$f(x) = \exp\left(\tan \tfrac{1}{2}x\right)$$

$$f'(x) = \sec^2 \tfrac{1}{2}x \left(\tfrac{1}{2}\right) \exp\left(\tan \tfrac{1}{2}x\right) \qquad \textbf{1,1,1}$$

$$= \tfrac{1}{2} \sec^2 \tfrac{1}{2}x \exp\left(\tan \tfrac{1}{2}x\right)$$

(b)

$$g(x) = \left(x^3 + 1\right) \ln\left(x^3 + 1\right)$$

$$g'(x) = 3x^2 \ln\left(x^3 + 1\right) + \left(x^3 + 1\right) \frac{3x^2}{x^3 + 1} \qquad \textbf{1,1}$$

$$= 3x^2 \ln\left(x^3 + 1\right) + 3x^2 \qquad \textbf{1}$$

B2.

$$A^2 - A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \qquad \textbf{M1}$$

$$= \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \textbf{1}$$

$$= 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textbf{1}$$

B3.

$$x = 0 \Leftrightarrow 5t^2 - 5 = 0 \Leftrightarrow t = \pm 1 \qquad \textbf{M1}$$

$$y = -3 \Leftrightarrow 3t^3 = -3 \Leftrightarrow t = -1$$

At $(0, -3)$, $t = -1$. **1**

$$\frac{dx}{dt} = 10t; \ \frac{dy}{dt} = 9t^2; \qquad \textbf{1}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t}{10} \qquad \textbf{1}$$

So when $t = -1$, the gradient is $-\frac{9}{10}$. **1**

B4.

$$\left(2a - \frac{3}{a}\right)^4 = (2a)^4 + 4(2a)^3\left(-\frac{3}{a}\right) + 6(2a)^2\left(-\frac{3}{a}\right)^2 + 4(2a)\left(-\frac{3}{a}\right)^3 + \left(-\frac{3}{a}\right)^4 \qquad \textbf{1 powers} \quad \textbf{1 coeff}$$

$$= 16a^4 - 96a^2 + 216 - \frac{216}{a^2} + \frac{81}{a^4} \qquad \textbf{1}$$

B5.

$$\frac{x^2 + 3}{x\left(1 + x^2\right)} = \frac{A}{x} + \frac{Bx + C}{1 + x^2} \qquad \textbf{M1}$$

$$x^2 + 3 = A\left(1 + x^2\right) + (Bx + C)x$$

$$x = 0 \quad \Rightarrow \quad 3 = A \qquad \textbf{1}$$

$$x^2 + 3 = 3\left(1 + x^2\right) + (Bx + C)x$$

$$x = 1 \quad \Rightarrow \quad 4 = 6 + B + C$$

$$x = -1 \quad \Rightarrow \quad 4 = 6 + B - C$$

$$2C = 0 \Rightarrow C = 0 \text{ and } B = -2 \qquad \textbf{1}$$

$$\int_{1/2}^{1} \frac{x^2 + 3}{x\left(1 + x^2\right)}\, dx = \int_{1/2}^{1} \frac{3}{x} - \frac{2x}{1 + x^2}\, dx$$

$$= \left[3 \ln x - \ln\left(1 + x^2\right)\right]_{1/2}^{1} \qquad \textbf{1}$$

$$= \left[0 - \ln 2\right] - \left[3 \ln \tfrac{1}{2} - \ln \tfrac{5}{4}\right] \qquad \textbf{1}$$

$$= \ln\left(\tfrac{5}{4} \times \tfrac{8}{1} \times \tfrac{1}{2}\right)$$

$$= \ln 5 \approx 1.609 \qquad \textbf{1}$$

B6. (a) *Method 1 − separating the variables*

$$\sin x \frac{dy}{dx} - 2y\cos x = 0$$

$$\frac{dy}{dx} = 2\frac{\cos x}{\sin x}y$$

$$\int \frac{dy}{y} = 2\int \frac{\cos x}{\sin x}\,dx \qquad \textbf{M1}$$

$$\ln y = 2\ln(\sin x) + C \qquad \textbf{1}$$

$$= \ln\left(\sin^2 x\right) + C \qquad \textbf{1}$$

$$y = \exp\left(C + \ln(\sin^2 x)\right) \qquad \textbf{1}$$

$$= e^C \sin^2 x$$

Method 2 − using an integrating factor

$$\sin x \frac{dy}{dx} - 2y\cos x = 0$$

$$\frac{dy}{dx} - 2\frac{\cos x}{\sin x}y = 0 \qquad \textbf{1}$$

$$\int -2\frac{\cos x}{\sin x}\,dx = -2\ln(\sin x) = \ln\left(\sin^{-2} x\right) \qquad \textbf{1}$$

Integrating factor = $\exp\left[\ln\left(\sin^{-2} x\right)\right] = \sin^{-2} x$

$$\frac{1}{\sin^2 x}\frac{dy}{dx} + \frac{-2\cos x}{\sin^3 x}y = 0 \qquad \textbf{1}$$

$$\frac{d}{dx}\left(\frac{y}{\sin^2 x}\right) = 0$$

$$y = A\sin^2 x \qquad \textbf{1}$$

(b)

$$\sin x \frac{dy}{dx} - 2y\cos x = 3\sin^3 x$$

$$\frac{dy}{dx} - 2\frac{\cos x}{\sin x}y = 3\sin^2 x. \qquad \textbf{1}$$

Integrating factor is

$$\exp\left(\int -2\frac{\cos x}{\sin x}\,dx\right) = \exp\left(-2\ln(\sin x)\right) = \frac{1}{\sin^2 x} \qquad \textbf{M1,1}$$

$$\frac{1}{\sin^2 x}\frac{dy}{dx} + \frac{-2\cos x}{\sin^3 x}y = 3 \qquad \textbf{1}$$

$$\frac{d}{dx}\left(\frac{y}{\sin^2 x}\right) = 3$$

$$\frac{y}{\sin^2 x} = 3x + D$$

$$y = (3x + D)\sin^2 x \qquad \textbf{1}$$

[END OF MARKING INSTRUCTIONS]