X204/701

NATIONAL
QUALIFICATIONS
2006

MONDAY, 22 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

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Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

Candidates should observe that $g\,\text{m s}^{-2}$ denotes the magnitude of the acceleration due to gravity.
Where appropriate, take its magnitude to be $9{\cdot}8\,\text{m s}^{-2}$.

A1. Relative to a rectangular coordinate system, the position of an ice skater at time t seconds is

$$\mathbf{r}(t) = \left(\frac{1}{3}t^3 - 4t^2\right)\mathbf{i} - (2t^2 - 1)\mathbf{j},$$

where $\mathbf{i}$, $\mathbf{j}$ are the unit vectors in the x, y directions respectively and distances are measured in metres.

Find the speed of the ice skater at the instant when the acceleration is parallel to the y-axis. **5**

A2. A piston oscillates about the point O with simple harmonic motion of amplitude $0{\cdot}25$ m.

Calculate the distance of the piston from O when its speed is half its maximum speed. **5**

A3. A lift is initially at rest at ground level. It begins to accelerate upwards at $\frac{1}{8}g\,\text{m s}^{-2}$. At the same instant, a light bulb in the ceiling of the lift begins to fall towards the lift floor. The initial distance between the lift floor and the light bulb is 2 metres.

(*a*) Measuring distances in metres relative to the ground level, show that the position of the light bulb relative to the lift floor is

$$\left(2 - \frac{9}{16}gt^2\right)\mathbf{j},$$

where $\mathbf{j}$ is the unit vector in the upward vertical direction, and t is the time in seconds from the start of the motion of the lift. **3**

(*b*) Calculate the distance the light bulb falls before hitting the lift floor. **3**

A4. A golfer strikes a golf ball from O across a horizontal section of ground, giving the ball an initial speed of $V\ \text{m s}^{-1}$ at an angle α to the horizontal.

(*a*) Show that the range, R metres, of the golf ball is given by

$$R = \frac{V^2}{g}\sin 2\alpha.$$ **4**

(*b*) The golfer intends the ball to land between two points A and B on the horizontal section such that $OA = L$ metres, $OB = 2L$ metres and OAB is a straight line.

Given that the angle of projection of the ball is $15°$, show that the initial speed must satisfy

$$\sqrt{2} < \frac{V}{\sqrt{gL}} < 2.$$ **3**

A5. A railway truck of mass $3m$ kilograms travelling at $u\ \text{m s}^{-1}$ along a straight horizontal track, collides and couples with a stationary truck of mass m kilograms. Due to the action of a constant resistive force of magnitude R newtons, the two trucks come to rest T seconds after the collision.

(*a*) Determine an expression for R in terms of m, u and T. **4**

(*b*) Find an expression, in terms of m and u, for the work done by R in bringing the trucks to rest. **3**

A6. A conical pendulum consists of a bobbin of mass m kilograms attached to one end, B, of a light elastic string AB of natural length l metres and modulus of elasticity $8mg$ newtons. The other end, A, of the string is held fixed. The bobbin moves in a horizontal circle with centre vertically below A, such that the angle between the string AB and the vertical is $45°$.

(*a*) Determine, in terms of l, the extension of the string beyond its natural length. **3**

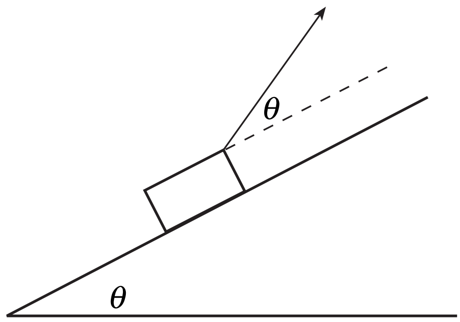
(*b*) Show that the angular speed, ω radians per second, of the bobbin is given by

$$\omega^2 = \frac{8g}{(1+4\sqrt{2})l}.$$ **3**

[Turn over

Marks

A7. Alan pulls a container with weight of magnitude W newtons at a constant speed up a rough plane, with coefficient of friction μ, inclined at an acute angle θ to the horizontal by means of a light inextensible rope, as shown below. The rope also makes an angle θ to the inclined plane.



(*a*) Show that the magnitude of the tension in the rope is given by

$$\left(\frac{\tan\theta + \mu}{1 + \mu\tan\theta}\right)W \text{ newtons.}$$ **6**

(*b*) Determine the range of values of θ for which the tension in the rope is less than the weight of the container. **4**

A8. A mass m kilograms is attached to one end, A, of a light inextensible string of length L metres, the other end of which is fixed at a point O. Initially the mass hangs vertically below O with the string taut. The mass is then given a horizontal speed of $\sqrt{\frac{7}{2}gL}$ m s^{-1}, causing it to start to travel in a vertical circle of centre O. Subsequently, the string OA makes an angle θ with the downward vertical through O.

(*a*) When $\theta = 45°$, find expressions for:

(i) the speed of the mass in terms of L and g; **4**

(ii) the magnitude of the tension in the string, in terms of m and g. **3**

(*b*) Determine the value of θ at which the string first becomes slack. **4**

Marks

A9. A box of mass m kilograms is dropped from rest from a point A at a height h metres above ground level. The box experiences a force of magnitude mkv^2 newtons due to air resistance, where v m s^{-1} is the speed of the box and k is a constant.

(*a*) Show that the speed of the box satisfies the differential equation

$$v\frac{dv}{dx} = g - kv^2,$$

where x metres is the distance fallen by the box from A. **2**

(*b*) By making the substitution $w = g - kv^2$, or otherwise, solve the differential equation in (*a*) to show that

$$v^2 = \frac{g}{k}(1 - e^{-2kx}).$$ **6**

(*c*) Given that $kh = 2$, calculate what fraction of the initial potential energy is used in doing work against the resistive force during the descent of the box to the ground. **3**

[*END OF SECTION A*]

[Turn over for Section B on *Page six*

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Calculate A^{-1} where $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}$.

Hence solve the system of equations

$$\begin{aligned} x + y &= 1 \\ 2x + 3y + z &= 2 \\ 2x + 2y + z &= 1. \end{aligned}$$

5

B2. Given that $y = \ln(1 + \sin x)$, where $0 < x < \frac{\pi}{2}$, show that $\dfrac{d^2y}{dx^2} = \dfrac{-1}{1 + \sin x}$. **5**

B3. Define $S_n = \sum_{r=1}^{n} r^2$, $n \geq 1$. Write down formulae for S_n and S_{2n+1}. **2**

Obtain a formula for $2^2 + 4^2 + \ldots + (2n)^2$. **1**

B4. Solve the differential equation

$$\cos^2 x \frac{dy}{dx} = y,$$

given that $y > 0$ and that $y = 2$ when $x = 0$. **5**

B5. Use the substitution $1 + x^2 = u$ to obtain $\displaystyle\int \frac{x^3}{\sqrt{1 + x^2}}\, dx$. **5**

B6. (*a*) Evaluate $\int_0^1 xe^{2x} dx$. **4**

(*b*) Use part (*a*) to evaluate $\int_0^1 x^2 e^{2x} dx$. **3**

(*c*) Hence obtain $\int_0^1 (3x^2 + 2x)e^{2x} dx$. **2**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]