2007 Applied Mathematics

Advanced Higher – Mechanics

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question B6, M1 means a method mark for using the partial fractions to work out the are. E is shorthand for error. For example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Applied Mathematics 2007
Section A – Mechanics

A1.

$$\mathbf{r} = (3t^2 - 12t + 5)\mathbf{i} + (4t - t^2)\mathbf{j}$$

$$\Rightarrow \mathbf{v} = (6t - 12)\mathbf{i} + (4 - 2t)\mathbf{j} \qquad \mathbf{1}$$

$$= 6(t - 2)\mathbf{i} - 2(t - 2)\mathbf{j}.$$

So $\mathbf{v} = 0$ when $t = 2$. **1**

At this time $\mathbf{r} = -7\mathbf{i} + 4\mathbf{j}$ **1**

and hence the distance from the origin is

$$|\mathbf{r}| = \sqrt{(-7)^2 + 4^2}$$

$$= \sqrt{65}\,(\approx 8{\cdot}1)\text{ m} \qquad \mathbf{1}$$

A2. Using $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{4\pi} = \dfrac{1}{2}$, **1**

and from $v^2 = \omega^2(a^2 - x^2)$ we obtain **1M**

$$0{\cdot}8^2 = \tfrac{1}{4}(a^2 - 0{\cdot}2^2) \qquad \mathbf{1}$$

$$a^2 = 2{\cdot}6$$

Hence, the amplitude $= \sqrt{2.6}\,(\approx 1{\cdot}61)$ m. **1**

A3. Momentum before $= 3mU - 2mU$

$= mU$

Momentum after $= 3mV + 2mU$, where V m s^{-1} is the speed of A after the collision

$= 3mV + 2mU$ **1**

By conservation of momentum

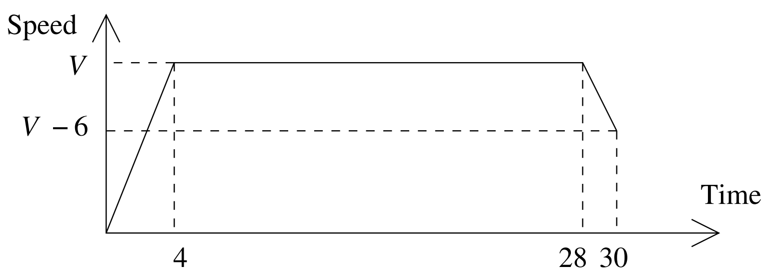
$$3mV + 2mU = mU \qquad \mathbf{1}$$

$$V = -\tfrac{1}{3}U$$

$$< 0 \qquad \mathbf{1}$$

i.e. the speed of A is $\tfrac{1}{3}U$ and its direction is reversed.

A4.



1

Using Area under graph = distance travelled

$$\tfrac{1}{2} \times 4 \times V + 24V + \tfrac{1}{2} \times 2 \times 6 + 2 \times (V - 6) = 200 \qquad \mathbf{2E1}$$

$$2V + 24V + 6 + 2V - 12 = 200$$

$$28V = 206$$

$$V = \frac{103}{14}\,(\approx 7{\cdot}4) \qquad \mathbf{1}$$

A5. Let the height of the tree be h metres.

Using $x = (V \cos\theta)t$ gives

$$70 = 30 \times 3 \cos\theta^\circ \qquad \textbf{1M}$$

$$\cos\theta^\circ = \frac{7}{9} \Rightarrow \theta \approx 38{\cdot}9 \qquad \textbf{1}$$

then, from $y = (V \sin\theta^\circ)t - \frac{1}{2}gt^2$

$$h = 30 \sin\theta^\circ \times 3 - \tfrac{1}{2} \times 9{\cdot}8 \times 3^2 \approx 12{\cdot}5 \qquad \textbf{1M}$$

Thus the height of the tree is approximately 12·5 metres. **1**

A6. Driving force = 500 N.

Speed = $\dfrac{P}{F} = \dfrac{15000}{500} = 30 \text{ m s}^{-1}$ **1M,1**

On the incline, driving force $= 500 + mg \sin\theta^\circ$

$= 500 + 1400g \sin\theta^\circ$ **1**

Using $P = FV$

$$500 + 1400g \sin\theta^\circ = \frac{15000}{15} \qquad \textbf{1}$$

$$\sin\theta^\circ = \frac{5}{14g}$$

$$\theta \approx 2{\cdot}1 \qquad \textbf{2E1}$$

A7. Using $v^2 = u^2 + 2as$

$$4^2 = 20a \qquad \textbf{M1}$$

$$a = \frac{16}{20} = 0{\cdot}8 \qquad \textbf{1}$$

Reaction force $R = mg \cos\theta^\circ = 2g \times \frac{\sqrt{3}}{2} = \sqrt{3}g$. **1**

Resolving parallel to the plane, Newton II gives

$$ma = mg \sin 30^\circ - \mu R \qquad \textbf{1}$$

$$1.6 = g - \mu(\sqrt{3}g) \qquad \textbf{1}$$

$$\mu = \frac{g - 1.6}{\sqrt{3}g}$$

$$\approx 0.48 \qquad \textbf{1}$$

Alternative method

Using the work-energy principle

Kinetic energy $E_K = \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times 4^2 = 16$ J. **1**

$$\text{Reaction force} = mg \cos 30^\circ$$

$$= \sqrt{3}g \text{ N} \qquad \textbf{1}$$

The work-energy principle gives

$$\text{work} + E_K = \text{gravitational PE} \qquad \textbf{1M}$$

$$\mu(\sqrt{3}g) \times 10 + 16 = 2g \times 10 \sin 30^\circ \qquad \textbf{1,1}$$

$$\mu = 0.48 \qquad \textbf{1}$$

A8. Let $\mathbf{v}_C$ and $\mathbf{v}_M$ be the velocities of the car and the motorcycle respectively.

$$\mathbf{v}_C = u\mathbf{j} \qquad \mathbf{v}_M = 2u\mathbf{i} \qquad \mathbf{1}$$

$$\mathbf{r}_C = (ut - 40)\mathbf{j} \qquad \mathbf{r}_M = 2ut\mathbf{i} \qquad \mathbf{1}$$

$$_C\mathbf{r}_M = (ut - 40)\mathbf{j} - 2ut\mathbf{i} \qquad \mathbf{1}$$

The distance D between M and C is given by

$$D^2 = (2ut)^2 + (ut - 40)^2 \qquad \mathbf{1}$$

$$= 5u^2t^2 - 80ut + 1600$$

$$\frac{d(D^2)}{dt} = 10u^2t - 80u \qquad \mathbf{1}$$

For a minimum

$$\frac{d(D^2)}{dt} = 0 \Rightarrow t = \frac{8}{u} \text{ i.e. } ut = 8 \qquad \mathbf{1}$$

so

$$D^2 = 5 \times 64 - 80 \times 8 + 1600$$

$$= 1280$$

$$D \approx 35.78 \qquad \mathbf{1}$$

So the minimum distance between M and C is 35.8 m.

A9. (a) Using v_R for the speed at R, by conservation of energy

$$\tfrac{1}{2}mv_R^2 + mg(2a) = \tfrac{1}{2}m\left(3\sqrt{ga}\right)^2 \qquad \mathbf{M1,1}$$

$$v_R^2 + 4ga = 9ga$$

$$v_R^2 = 5ga$$

$$v_R = \sqrt{5ga} \qquad \mathbf{1}$$

(b) For the car to be in contact with the track at R

$$\frac{mv_R^2}{a} \geqslant mg \qquad \mathbf{1}$$

Now

$$\frac{mv_R^2}{a} = \frac{m\,(5ga)}{a} = 5mg > mg \qquad \mathbf{1}$$

(c) The time to return to PQ satisfies

$$2a = \tfrac{1}{2}gt^2$$

$$t = \sqrt{\frac{4a}{g}} \qquad \mathbf{1}$$

Thus the distance SQ is given by

$$SQ = \sqrt{5ga} \times \sqrt{\frac{4a}{g}} \qquad \mathbf{1}$$

$$= \sqrt{5}\,(2a) \qquad \mathbf{1}$$

$$= 2\sqrt{5}a$$

(d) At S, the vertical component of the speed is

$$V = \sqrt{4ga} \qquad \mathbf{1}$$

Hence

$$\tan \phi^\circ = \frac{\sqrt{4ga}}{\sqrt{5ga}} \qquad \mathbf{1}$$

$$= \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

A10. (a) By Newton II

$$ma = \text{weight} - \text{buoyancy} - \text{resistance}$$

$$mv\frac{dv}{dx} = mg - 3mg - 2mv^2 \qquad \mathbf{1,1}$$

$$v\frac{dv}{dx} = -2\left(g + v^2\right)$$

(b) Separating the variables and integrating gives

$$\int \frac{v\,dv}{v^2 + g} = \int (-2)\ dx \qquad \mathbf{1}$$

$$\tfrac{1}{2}\ln\left(v^2 + g\right) = -2x + c \qquad \mathbf{1}$$

When $x = 0, v = U \Rightarrow c = \tfrac{1}{2}\ln\left(U^2 + g\right)$ so **1**

$$\ln\left(v^2 + g\right) = -4x + \ln\left(U^2 + g\right) \qquad \mathbf{1}$$

$$\ln\left(\frac{v^2 + g}{U^2 + g}\right) = -4x \qquad \mathbf{1}$$

$$v^2 + g = \left(U^2 + g\right)e^{-4x}$$

(c) When $v = 0$, **1**

$$e^{4x} = \frac{4.9^2 + 9.8}{9.8} \qquad \mathbf{1}$$

$$= 3.45$$

$$4x = \ln 3.45$$

$$x = \tfrac{1}{4} \times 1.238\ldots \approx 0.31 \qquad \mathbf{1}$$

The greatest depth is approximately 31 cm.

A11. (a) Balancing the gravitational forces gives

$$\frac{G(4M)m}{x^2} = \frac{GMm}{(L - x)^2} \qquad \mathbf{1,1}$$

$$4\left(L^2 - 2Lx + x^2\right) = x^2 \qquad \mathbf{1}$$

$$3x^2 - 8Lx + 4L^2 = 0$$

$$(3x - 2L)(x - 2L) = 0 \qquad \mathbf{1}$$

$$\text{i.e.}\, x = \frac{2L}{3} \text{ or } x = 2L$$

$$\text{So}\, x = \frac{2L}{3} \quad (\text{since}\, x < L) \qquad \mathbf{1}$$

(b) Let the angular speed be ω. Then, using

$$m(3R)\omega^2 = \frac{G(4M)m}{(3R)^2} \qquad \mathbf{M1,1}$$

$$\omega^2 = \frac{4GM}{27R^3} \qquad \mathbf{1}$$

But $\omega = \dfrac{2\pi}{T}$ so

$$\frac{4\pi^2}{T^2} = \frac{4GM}{27R^3} \qquad \mathbf{1}$$

$$T^2 = \frac{27\pi^2 R^3}{GM} \qquad \mathbf{1}$$

$$T = \pi\sqrt{\frac{27R^3}{GM}}$$

Section B – Mathematics for Applied Mathematics

B1.

$$\int_0^{\pi/6} x \sin 3x \, dx = \left[x \int \sin 3x \, dx - \int 1 . \left(-\frac{1}{3} \cos 3x \right) \right]_0^{\pi/6} \quad \textbf{2E1}$$

$$= \left[x \left(-\frac{1}{3} \cos 3x \right) + \frac{1}{9} \sin 3x \right]_0^{\pi/6} \quad \textbf{1}$$

$$= -\frac{\pi}{18} \cos \frac{\pi}{2} + \frac{1}{9} \sin \frac{\pi}{2} - (0 + 0) \quad \textbf{1}$$

$$= \frac{1}{9} \quad \textbf{1}$$

B2.

$$\left(x^3 - \frac{2}{x} \right)^4 = \left(x^3 \right)^4 + 4\left(x^3 \right)^3 \left(-\frac{2}{x} \right) + 6\left(x^3 \right)^2 \left(-\frac{2}{x} \right)^2 + 4x^3 \left(-\frac{2}{x} \right)^3 + \left(-\frac{2}{x} \right)^4 \quad \textbf{2E1}$$

$$= x^{12} - 8x^8 + 24x^4 - 32 + \frac{16}{x^4} \quad \textbf{2E1}$$

B3.

$$x = \frac{t}{t^2 + 1} \Rightarrow \frac{dx}{dt} = \frac{1\left(t^2 + 1\right) - t\,(2t)}{\left(t^2 + 1\right)^2} = \frac{1 - t^2}{\left(t^2 + 1\right)^2} \quad \textbf{2E1}$$

$$y = \frac{t - 1}{t^2 + 1} \Rightarrow \frac{dy}{dt} = \frac{1\left(t^2 + 1\right) - (t - 1)(2t)}{\left(t^2 + 1\right)^2} = \frac{1 + 2t - t^2}{\left(t^2 + 1\right)^2} \quad \textbf{2E1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1 + 2t - t^2}{(t^2 + 1)^2}}{\frac{1 - t^2}{(t^2 + 1)^2}} \quad \textbf{1}$$

$$= \frac{1 + 2t - t^2}{1 - t^2}$$

B4.

$$A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix}$$

$$\det A = \lambda(\lambda - 3) - 4 \quad \textbf{1}$$

A matrix is singular when its determinant is 0.

$$\lambda^2 - 3\lambda - 4 = 0 \quad \textbf{1}$$

$$(\lambda + 1)(\lambda - 4) = 0$$

$$\lambda = -1 \text{ or } \lambda = 4 \quad \textbf{1}$$

When $\lambda = 3, A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$ so $A^{-1} = \frac{1}{-4} \begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix}$. **1**

B5.

$$x\frac{dy}{dx} - y = x^2 e^x$$

$$\frac{dy}{dx} - \frac{1}{x}y = xe^x \qquad \mathbf{1}$$

Integrating factor:

$$\exp\left(\int \frac{-1}{x}\,dx\right) \qquad \mathbf{1}$$

$$= \exp(-\ln x) = x^{-1} \qquad \mathbf{1}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = e^x$$

$$\frac{y}{x} = \int e^x dx = e^x + c \qquad \mathbf{1}$$

$$\therefore \quad y = x(e^x + c)$$

$y = 2$ when $x = 1 \Rightarrow 2 = e + c$

$$\Rightarrow y = x(e^x - e + 2) \qquad \mathbf{1}$$

B6.

$$\frac{8}{x(x+2)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4} \qquad \mathbf{1}$$

$$8 = A(x+2)(x+4) + Bx(x+4) + Cx(x+2)$$

$$x = 0 \Rightarrow 8A = 8 \Rightarrow A = 1 \qquad \mathbf{1}$$

$$x = -2 \Rightarrow -4B = 8 \Rightarrow B = -2 \qquad \mathbf{1}$$

$$x = -4 \Rightarrow 8C = 8 \Rightarrow C = 1 \qquad \mathbf{1}$$

$$\frac{8}{x(x+2)(x+4)} = \frac{1}{x} + \frac{-2}{x+2} + \frac{1}{x+4}$$

$$\text{Area} = \int_1^2 \left(\frac{1}{x} + \frac{-2}{x+2} + \frac{1}{x+4}\right)dx \qquad \mathbf{1M}$$

$$= [\ln x - 2\ln(x+2) + \ln(x+4)]_1^2 \qquad \mathbf{1}$$

$$= \left[\ln\frac{x(x+4)}{(x+2)^2}\right]_1^2$$

$$= \ln\frac{12}{16} - \ln\frac{5}{9} = \ln\frac{12}{16} \times \frac{9}{5} = \ln\frac{27}{20} \qquad \mathbf{3E1}$$

[END OF MARKING INSTRUCTIONS]