Mathematics
Mechanics 1
Advanced Higher

7548

Summer 2000

HIGHER STILL

Mathematics

Mechanics 1

Advanced Higher

Support Materials

MECHANICS 1 (ADVANCED HIGHER)

Introduction

These support materials for Mathematics were developed as part of the Higher Still Development Programme in response to needs identified at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics* (SEED 1999) and in the Mathematics Subject Guide.

These notes are intended to support teachers/lecturers in the teaching of Mechanics 1 (AH). The resources referred to within the material are:

Understanding Mechanics
Sadler & Thorning Oxford Univ. Press 019 914675 6 [abbreviated to **S&T**]

Mechanics
R. C. Solomon, John Murray, 0719570824 [abbreviated to **RCS**]

Mechanics
Ted Graham, Collins Educational, 000322372 8 [abbreviated to **TG**]

Mathematics- Mechanics and Probability
Bostock & Chandler, Stanley Thornes, 0859501418 [abbreviated to **B&C**]

The Complete A-Level Mathematics
Orlando Gough Heinemann Educational Books 0435513451 [abbreviated to **OG**]

MECHANICS 1 (AH): AN OVERVIEW

This unit contains much of the work of CSYS Paper V, sections 1 – 4, with some omissions which are in Mechanics 2 (AH). This ensures that the mathematics required does not go beyond that of Mathematics 1(H) and Mathematics 2(H) with the exception of vectors. However sufficient time is allowed in this unit to incorporate the teaching of the necessary vector work and Mechanics offers an excellent approach to this topic.

Many Mechanics and A-Level Mathematics textbooks will provide ample work and examples for this unit. A selection has been chosen for illustration of these notes, namely:

Throughout Mechanics the use of the correct units is very important. Some textbooks make some general comments about units in an introductory chapter. Others incorporate units as the quantities arise.

The order in which mechanics textbooks deal with topics varies considerably, and this is particularly the case for the work of this unit.

Of the textbooks listed, the four mechanics books first listed also cover most, if not all, of the content of Mechanics 2(AH) and Orlando Gough also covers a substantial part of it.

MOTION IN A STRAIGHT LINE

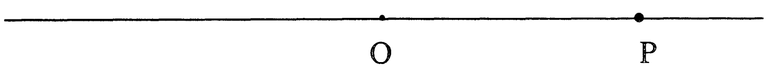
CONTENT
know the meaning of position, displacement, velocity, acceleration, uniform speed, uniform acceleration, scalar quantity, vector quantity.
draw, interpret and use distance/time, velocity/time and acceleration/time graphs
know that the area under a velocity/time graph represents the distance travelled
know that the rates of change $v = \frac{dx}{dt} = \dot{x}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d\dot{x}}{dt} = \dot{v} = \ddot{x}$
solve problems involving motion in one dimension where the acceleration is dependent on time, i.e. $a = \frac{dv}{dt} = f(t)$

Comments

Students should be very aware of the distinction between scalar and vector quantities, particularly in the case of speed and velocity and also distance and displacement.

Teaching Notes

MOTION IN A STRAIGHT LINE



Students should know that if a particle P is moving in a straight line along the x-axis and $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis then:

The **displacement** of P from O at any instant is the vector $\boldsymbol{x} = x\boldsymbol{i}$, where x = OP.
If P is to the right of O, x is positive and if P is to the left of O, x is negative.

The **velocity** of the particle is the rate of change of its displacement and is the vector $\boldsymbol{v} = \frac{dx}{dt}\boldsymbol{i}$.

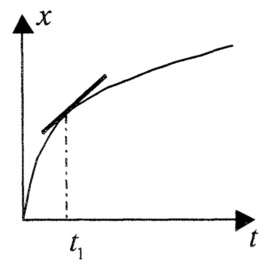
If P is moving to the right $\frac{dx}{dt}$ is positive

and if P is moving to the left $\frac{dx}{dt}$ is negative.

If $|\,\boldsymbol{v}\,|$ is denoted by v then $v = \frac{dx}{dt}$ and

$x = \int v\,dt$.

$\frac{dx}{dt}$ is sometimes denoted by $\dot{x}$.



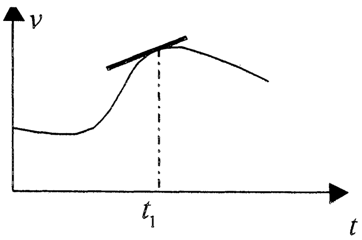
gradient of graph at $t = t_1$
$= \frac{dx}{dt}$ = **velocity** at $t = t_1$

The **acceleration** of the particle is the rate of change of its velocity and is the vector $\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{i}$.

If the velocity is increasing $\frac{dv}{dt}$ is positive and if the velocity is decreasing $\frac{dv}{dt}$ is negative.

If $|\,\boldsymbol{a}\,|$ is denoted by a then $a = \frac{dv}{dt}$ and $v = \int a\,dt$

Also $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ and $\frac{d^2x}{dt^2}$ is sometimes denoted by $\ddot{x}$.



gradient of graph at $t = t_1$

$= \frac{dv}{dt}$ = **acceleration**

area under graph $t = 0$ to $t = t_1$

$= \int_0^{t_1} v\,dt$ = **displacement** at $t = t_1$

The **area under an acceleration/time graph** from $t = 0$ to $t = t_1$ is

$$\int_0^{t_1} a\,dt = \textbf{velocity} \text{ at } t = t_1$$

Note

Students should be familiar with the dot notation for differentiating with respect to time. This is not used in all textbooks. Of the textbooks listed S&T, RCS and B&C use this notation while the others do not.

WORKED EXAMPLES

Example 1

A body moves along the x-axis with velocity, measured in ms^{-1}, given by

$$v = (3t^2 - 18t + 15)\boldsymbol{i},$$

where $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis and t is the time in seconds from the start of the motion.

At the start of the motion the displacement of the body from the origin is 30 m. Find:

a) the initial speed of the body;

b) the values of t for which the body is at rest;

c) the acceleration of the body when $t = 6$;

d) the displacement of the body from O when $t = 3$.

Solution

a) $v = 3t^2 - 18t + 15$. When $t = 0$, $v = 15$ ms^{-1}

b) $v = 3(t^2 - 6t + 5) = 3(t - 1)(t - 5) = 0$ when $t = 1, 5$

$\therefore$ body is at rest after 1 second and after 5 seconds from the start

c) $a = \dfrac{dv}{dt} = 6t - 18$ When $t = 6$, $a = 36 - 18 = 18$ ms^{-1}

d) $x = \int v dt = t^3 - 9t^2 + 15t + c$ Now $x = 30$ when $t = 0$ so $c = 30$

$x = t^3 - 9t^2 + 15t + 30$ When $t = 3$, $x = 27 - 81 + 45 + 30$ $\therefore$ displacement from O is 21 m in the positive direction.

Example 2

A body moves along the x-axis from rest at the origin with acceleration, measured in ms^{-2}, given by

$$\boldsymbol{a} = (2 - \sqrt{t})\boldsymbol{i}$$

where $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis and t is the time in seconds from the start of the motion.

a) Show that its speed increases to a maximum value and then decreases.

b) Find

i) the time till the body is instantaneously at rest again

ii) the time before it again passes through its starting point

Solution

a) $\dfrac{dv}{dt} = a = (2 - \sqrt{t})$

for $0 < t < 4$, $a > 0$ and so v is increasing. When $t = 4$, v has a stationary value and for $t > 4$, v is decreasing. At $t = 4$, v has a maximum value

$v = \int a dt = \int (2 - \sqrt{t})dt = 2t - \dfrac{2}{3}t^{\frac{3}{2}} + c$ when $t = 0$, $v = 0$ thus $c = 0$

and so $v = 2t - \frac{2}{3}t^{\frac{3}{2}}$ when $t = 4$, $v = 8 - \frac{2}{3} \times 8 = \frac{8}{3}$

b)

i) Body is instantaneously at rest when $v = 0$, ie when $0 = 2t - \frac{2}{3}t^{\frac{3}{2}}$ i.e. when t = 0 or t = 9 and so is again at rest after 9 seconds

ii) $\frac{dx}{dt} = v = 2t - \frac{2}{3}t^{\frac{3}{2}} \Rightarrow x = t^2 - \frac{4}{15}t^{\frac{5}{2}} + c$

when $t = 0, x = 0$ so $c = 0$ and hence $x = t^2 - \frac{4}{15}t^{\frac{5}{2}}$

The body passes through its starting point when $x = 0$

i.e. when $t = 0$ or $t = \left(\frac{15}{4}\right)^2$.

Thus body passes through the starting point again after 14·06 seconds.

Example 3

A particle starts from rest at the origin and moves along the x-axis with acceleration, measured in ms^{-2}, given by $\boldsymbol{a} = (6 - 2t)\boldsymbol{i}$, where $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis and t is the time in seconds from the start of the motion.

Find the maximum speed of the particle. Find also the displacement of the particle on reaching maximum speed.

Solution

$\frac{dv}{dt} = a = 6 - 2t \Rightarrow v = 6t - t^2 + c$

$v = 0$ when $t = 0 \Rightarrow c = 0 \quad \Rightarrow v = 6t - t^2$

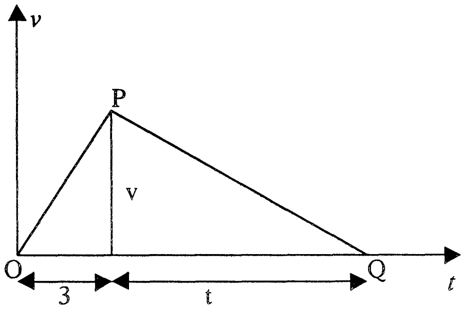
Maximum speed occurs when $a = 0$ ie when $t = 3$, hence max speed is 9ms^{-1}

$v = 6t - t^2 \Rightarrow x = 3t^2 - \frac{t^3}{3} + c$. But $x = 0$ when $t = 0$ $\therefore$ $c = 0$. Thus $x = 3t^2 - \frac{t^3}{3}$

At max speed, $t = 3 \Rightarrow$ displacement $= 3 \times 3^2 - \frac{3^3}{3} = 27 - 9 = 18$ metres.

Example 4

A particle starts from rest and moves in a straight line with uniform acceleration 6ms^{-2} for 3 seconds. It is then brought to rest with uniform acceleration of –2ms^{-2}. Draw a velocity/time graph and use it to find the distance travelled by the particle.



$m_{OP} = \frac{v}{3} = 6 \Rightarrow v = 18 \text{ ms}^{-1}$

$m_{PQ} = -2 \Rightarrow \frac{18}{t} = 2 \Rightarrow t = 9$ seconds

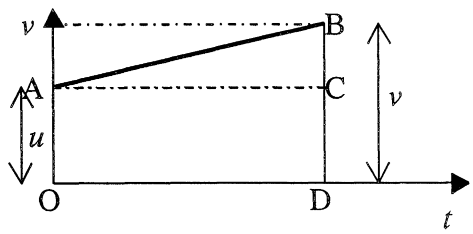
Distance = area OPQ

$= \frac{1}{2} \times 12 \times 18 = 108\text{m}$

Equations of Motion of a body moving with uniform acceleration

Consider a particle moving in a straight line with constant acceleration $\boldsymbol{a} = a\boldsymbol{i}$ with initial velocity $\boldsymbol{u} = u\boldsymbol{i}$ and final velocity $v = v\boldsymbol{i}$ after time t.

The velocity/time graph for this motion is as shown below.



Now $\quad a = \text{gradient of AB} = \frac{\text{CB}}{\text{AC}} = \frac{v-u}{t} \quad \Rightarrow a = \frac{v-u}{t} \quad \Rightarrow \boxed{v = u + at}$

Also x = area under the line = area of rectangle OACD + area of triangle ACB

Hence $x = ut + \frac{1}{2}(at)t = ut + \frac{1}{2}at^2$ We use x to denote displacement when motion is along the x-axis. Otherwise we normally use s. Thus $s = ut + \frac{1}{2}at^2$.

Note: In this unit students are expected to be able to derive these equations using calculus.

CONTENT
derive, by calculus methods, and use the equations governing motion in a straight line with constant acceleration, namely: $v = u + at,\ s = ut + \frac{1}{2}at^2$ and from these $v^2 = u^2 + 2as,\ s = \frac{(u+v)t}{2}$ solve analytically problems involving motion in one dimension under constant acceleration, including vertical motion under constant gravity.

Comments

Students need to appreciate that these equations are for motion with constant acceleration only. The general technique is to use calculus and this requires emphasis.

Teaching Notes

Not all textbooks give the proof by Calculus. Many establish them from the velocity/time graph as shown in the previous example.

Proof using calculus

acceleration = $\frac{dv}{dt} = a \quad \Rightarrow v = at + c_1$ (a constant)

If initial speed is u then $v = u$ when $t = 0$ so $c_1 = 0 \quad \therefore\ v = u + at$ (1)

Now, $v = u + at \Rightarrow \frac{ds}{dt} = u + at \ \Rightarrow s = ut + \frac{1}{2}at^2 + c_2$

If body starts at origin then $s = 0$ when $t = 0$, so $c_2 = 0$. Hence $s = ut + \frac{1}{2}at^2$ (2)

From (1) $t = \frac{v-u}{a}$ substituting this in (2) $s = \frac{u(v-u)}{a} + \frac{1}{2}\frac{(v-u)^2}{a}$

giving, after rearrangement $v^2 = u^2 + 2as$ (3)

From (3) $s = \frac{v^2 - u^2}{2a} = \frac{(v+u)(v-u)}{2a} = \frac{(u+v)t}{2}$ (4)

WORKED EXAMPLES

Example 1

A particle moving in a straight line with constant acceleration increases its velocity from $4ms^{-1}$ to $16ms^{-1}$ in 6 seconds. Find the constant acceleration and the distance travelled during the 6 seconds.

Solution

$v = u + at \Rightarrow 16 = 4 + 6a \Rightarrow a = 2ms^{-1}$. $v^2 = u^2 + 2as \Rightarrow 256 = 16 + 4s \Rightarrow s = 60m$

Example 2

A ball is thrown vertically upwards, with speed $7{\cdot}7\ ms^{-1}$, from the top of a sheer cliff of height 21m. Find

a) the time taken for the ball to reach the foot of the cliff;

b) the velocity of the ball at the instant it hits the ground.

Solution

a) Taking **upwards** as the positive direction, $u = 7{\cdot}7ms^{-1}$, $a = -\ 9{\cdot}8\ ms^{-2}$ and if the origin is the top of the cliff the displacement of the foot of the cliff is -21m.

Using $s = ut + \frac{1}{2}at^2$ we have $-21 = 7{\cdot}7t - 4{\cdot}9t^2 \Rightarrow 49t^2 - 77t - 21 = 0$

$\Rightarrow 7t^2 - 11t - 30 = 0 \Rightarrow (7t + 10)(t - 3) = 0$

$\Rightarrow t = -\frac{10}{7}$ or $t = 3$ but $t > 0$ $\therefore t = 3$

b) Using $v = u + at$ we have $v = 7{\cdot}7 - 9{\cdot}8 \times 3 = 7{\cdot}7 - 29{\cdot}4 = -21{\cdot}7$

Hence the velocity is $21{\cdot}7\ ms^{-1}$ vertically downwards

RESOURCES/EXAMPLES

S&T Chapter 16 (Use of Calculus)
Pages 390-392; 397; Ex 16A, N^os 1 – 14,44 - 48
Chapter 2 (Distance, Velocity and Acceleration)
Pages 32,33,36; Ex 2E N^os 1 – 6
Pages 24-32; Ex 2C (a selection); Ex 2D

RCS Chapter 2 (Kinematics)
Pages 30 – 37; Ex 2F, N^os 1–5;Ex 2G, N^os 1-5; Ex 2H
Pages 37,38; Ex 2I, N^os 1 – 5
Pages 23 – 30; Ex 2C; Ex 2D, 2E

TG Chapter 7 (Motion with variable forces and acceleration)
Pages 94, 96 example 7.2, 100,101
Pages 97-99; Ex 7.1A, N^os 3,4,6,7; Ex 7.1 B, N^os 2,3,4
Page 104,105; Ex 7.2 A N^os 2,3,6; Ex 7.2B, N^os 3,4,8
Page 106; consolidation Ex, N^os 1,3,5

Chapter 4 (Kinematics in one dimension)
Pages 45 – 52; Ex 4.1A, 4.1B; Ex 4.2A, N^os 1 - 7,9
Pages 53,55 Ex 4.2B, N^os 1,2,4,6,8

B&C Chapter 11 (General Motion of a Particle)
Pages 338-340; Ex 11a, N^os 1-6, 7,9, 10, 13
Chapter 4 (Velocity and Acceleration)
Pages 111, 112, 119-134;Ex 4e; Ex4f; Ex 4g

OG Chapter 7 (Kinematics of a Particle)
Pages 367-370; Ex 7.1:1, N^os 1,2,7,8,11-16
Page 372,373, Ex7.1:2, N^os 1-4, 6,11-17, 19 - 23, 27
Pages 376-379, Ex7.1:3 N^os 1-19; Ex 7.1:4

POSITION, VELOCITY AND ACCELERATION VECTORS INCLUDING RELATIVE MOTION

CONTENT
know the meaning of the terms relative position, relative velocity and relative acceleration, air speed, ground speed and nearest approach be familiar with the notation $\underline{r}_P$ for the position vector of P $\underline{v}_P = \dot{\underline{r}}_P$ for the velocity vector of P $\underline{a} = \dot{\underline{v}}_P = \ddot{\underline{r}}_P$ for the acceleration vector of P $\overrightarrow{PQ} = \underline{r}_Q - \underline{r}_P$ for the position vector of Q relative to P $\underline{v}_Q - \underline{v}_P = \dot{\underline{r}}_Q - \dot{\underline{r}}_P$ for the velocity of Q relative to P $\underline{a}_Q - \underline{a}_P = \dot{\underline{v}}_Q - \dot{\underline{v}}_P = \ddot{\underline{r}}_Q - \ddot{\underline{r}}_P$ for the acceleration of Q relative to P resolve vectors into components in two and three dimensions differentiate and integrate vector functions of time use position, velocity and acceleration vectors and their components in two and three dimensions; these vectors may be functions of time. apply position, velocity and acceleration vectors to solve practical problems, including problems on navigation of ships and aircraft and on the effect of winds and currents solve problems involving collisions and nearest approach

Comments

It is common practice, in writing mathematics, to underline letters representing vectors and in print to denote vectors using bold italic script. Notation can, however, vary from textbook to textbook. For example some may use ${}_Q\boldsymbol{v}_P$ in preference to $\boldsymbol{v}_Q - \boldsymbol{v}_P$

Functions to be differentiated will be within the scope of Mathematics 1(H) and Mathematics 2(H). Examples in textbooks involving trigonometric functions can be omitted.

Students should be able to solve problems involving relative velocities both by using trigonometric calculations in triangles and by vector components. Solutions by scale drawing would not be accepted.

Although students who have not met vectors before, eg in Mathematics 3(AH) or in Physics should be able to pick up what is required in the course of this unit they could benefit from a brief study of the resources used at Higher.

Assumed knowledge of vectors

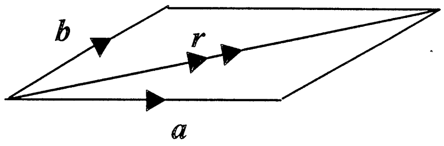
It is assumed that students:

- know the terms **position vector** and **unit vector**
- know the properties of **vector addition** and **multiplication of a vector by a scalar**
- know and can apply the **basis vectors** $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

Resolving Vectors

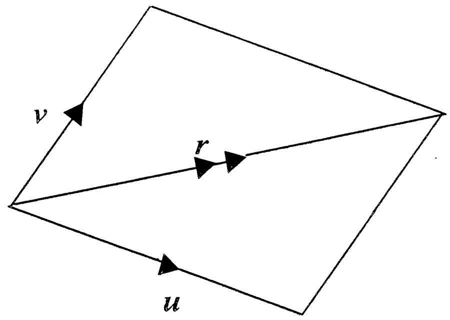
This is the opposite process to adding two vectors to obtain their sum. It provides the basis of much of the vector work in the latter part of this unit and teachers may wish to delay its introduction until it is needed

When **resolving a vector** we start with a single vector and replace it with two vectors.



$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{r}$ expresses the sum of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as a single vector $\boldsymbol{r}$ (the **resultant**)
$\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b}$ expresses $\boldsymbol{r}$ as the sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (its **components**)

$\boldsymbol{r}$ can be resolved into components in infinitely many ways. For example, in the diagram on the right $\boldsymbol{r} = \boldsymbol{u} + \boldsymbol{v}$



It is normal practice, in work in two dimensions to resolve a vector into two vectors at right angles **to one another.**

Components and the Sum of Vectors

Consider the perpendicular unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

Then, if $\boldsymbol{r}_1$ is any vector in the plane of $\boldsymbol{i}$ and $\boldsymbol{j}$, it can be expressed as

$$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j}$$

and if $\boldsymbol{r}_2$ is another vector in the plane of $\boldsymbol{i}$ and $\boldsymbol{j}$, it can be expressed as

$$\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j}$$

$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j}$ and $\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} \Rightarrow \boldsymbol{r}_1 + \boldsymbol{r}_2 = (x_1\boldsymbol{i} + y_1\boldsymbol{j}) + (x_2\boldsymbol{i} + y_2\boldsymbol{j})$.

Hence, $\boldsymbol{r}_1 + \boldsymbol{r}_2 = (x_1 + x_2)\boldsymbol{i} + (y_1 + y_2)\boldsymbol{j}$

This result can be extended to any number of vectors in the plane, and to any number of vectors in three dimensions. The component of the sum of a number of vectors in a particular direction is equal to the sum of the components of the individual vectors in that direction.

Differentiation and integration of vector functions of time

At Higher we considered the position vectors of fixed points. Here we deal with the position vectors of moving particles, or bodies, and so the components are functions of time.

Differentiation

If P is a particle then we denote the position vector of P at time t by $\boldsymbol{r}_P$

If $\boldsymbol{r}_P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then the velocity vector of P is $\boldsymbol{v}_P = \frac{d\boldsymbol{r}_P}{dt}$ where

$$\frac{d\boldsymbol{r}_P}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

Using the 'dot' notation for differentiation with respect to time this becomes

$$\dot{\boldsymbol{r}}_P = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$$

The acceleration vector of P is $\boldsymbol{a}_P = \frac{d\boldsymbol{v}_P}{dt} = \dot{\boldsymbol{v}}_P = \ddot{\boldsymbol{r}}_P = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$

Integration

Given its velocity (acceleration) vector we integrate to find the position (velocity) vector of the particle P.

For example:

Suppose the velocity vector of a particle, P, is given by $\boldsymbol{v}_P = 2t\boldsymbol{i} + \boldsymbol{j} + 6t\boldsymbol{k}$

then $\boldsymbol{r}_P = \int \boldsymbol{v}_P dt = (t^2 + c_1)\boldsymbol{i} + (t + c_2)\boldsymbol{j} + (3t^2 + c_3)\boldsymbol{k}$

$$= t^2\boldsymbol{i} + t\boldsymbol{j} + 3t^2\boldsymbol{k} + (c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k})$$

where $c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$ is a constant vector $\boldsymbol{c}$

In practice we simply write the result of the integration as $\boldsymbol{r}_P = t^2\boldsymbol{i} + t\boldsymbol{j} + 3t^2\boldsymbol{k} + \boldsymbol{c}$ and determine $\boldsymbol{c}$ from the initial conditions.

For example, suppose, that the particle started from the point with position vector $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ then $\boldsymbol{r}_p = 2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ when $t = 0$, and so $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ and hence $\boldsymbol{r}_P = (t^2 + 2)\boldsymbol{i} + (t - 1)\boldsymbol{j} + (3t^2 + 5)\boldsymbol{k}$

WORKED EXAMPLES

Example 1

A particle, P, moves from the point with position vector $-24\boldsymbol{i}-72\boldsymbol{j}+6\boldsymbol{k}$, with initial velocity $4\boldsymbol{i}+5\boldsymbol{k}$, and subject to an acceleration of $4\boldsymbol{j}-2\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the direction of orthogonal axes Ox, Oy and Oz respectively, and t is the time, in seconds, from the start of the motion. Distances are measured in metres and velocities in ms^{-1}.

Find an expression for the velocity of the particle at time t and calculate the speed of the particle when $t = 2$. Find the position vector of the particle at time t. When will the particle pass through the origin?

Solution

$\ddot{\boldsymbol{r}} = \boldsymbol{a}_P = 4\boldsymbol{j}-2\boldsymbol{k} \quad \Rightarrow \boldsymbol{v}_P = 4t\boldsymbol{j}-2t\boldsymbol{k}+\boldsymbol{c}_1$

Now $\boldsymbol{v}_P = 4\boldsymbol{i}+5\boldsymbol{k}$ when $t = 0$ so $\boldsymbol{c}_1 = 4\boldsymbol{i}+5\boldsymbol{k}$ hence $\boldsymbol{v}_P = 4\boldsymbol{i}+4t\boldsymbol{j}+(5-2t)\boldsymbol{k}$

When t = 2, $\boldsymbol{v}_P = 4\boldsymbol{i}+8\boldsymbol{j}+\boldsymbol{k}$

Speed = $|\boldsymbol{v}_P| = \sqrt{16+64+1} = \sqrt{81} = 9\text{ms}^{-1}$

$\dot{\boldsymbol{r}}_P = \boldsymbol{v}_P = 4\boldsymbol{i}+4t\boldsymbol{j}+(5-2t)\boldsymbol{k} \Rightarrow \boldsymbol{r}_P = 4t\boldsymbol{i}+2t^2\boldsymbol{j}+(5t-t^2)\boldsymbol{k}+\boldsymbol{c}_2$

Now $\boldsymbol{r}_P = -24\boldsymbol{i}-72\boldsymbol{j}+6\boldsymbol{k}$ when t = 0 so $\boldsymbol{c}_2 = -24\boldsymbol{i}-72\boldsymbol{j}+6\boldsymbol{k}$

$$\begin{aligned}\text{Hence } \boldsymbol{r}_P &= (4t-24)\boldsymbol{i}+(2t^2-72)\boldsymbol{j}+(6+5t-t^2)\boldsymbol{k}\\ &= 4(t-6)\boldsymbol{i}+2(t-6)(t+6)\boldsymbol{j}+(6-t)(1+t)\boldsymbol{k}\\ &= (t-6)[4\boldsymbol{i}+2(t+6)\boldsymbol{j}-(1+t)\boldsymbol{k}]\end{aligned}$$

Particle passes through the origin if and when $\boldsymbol{r}_P = \boldsymbol{0}$

Thus particle passes through origin after 6 seconds

Example 2

A particle, P, moves so that $\boldsymbol{r}_P = 4t\boldsymbol{i}+\frac{t^3}{4}\boldsymbol{j}$ for $0 \le t \le 6$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are perpendicular unit vectors and t is the time in seconds from the start of the motion.

Find the speed of the particle at the start and also when $t = 4$ (distances are measured in metres and velocities in ms^{-1}).

Find the acceleration of the particle at time t.

Solution

$\boldsymbol{r}_P = 4t\boldsymbol{i}+\frac{t^3}{4}\boldsymbol{j} \Rightarrow \boldsymbol{v}_P = \dot{\boldsymbol{r}}_P = 4\boldsymbol{i}+\frac{3t^2}{4}\boldsymbol{j}$

When t = 0, $\boldsymbol{v}_P = 4\boldsymbol{i}$ hence initial speed is 4ms^{-1}

When $t = 4$, $\quad v_P = 4i + 12j$ so speed = $|v_P| = \sqrt{4^2 + 12^2} = \sqrt{160} = 12.65\text{ms}^{-1}$

$\dot{r}_P = 4i + \frac{3t^2}{4}j \qquad \Rightarrow a_P = \ddot{r} = \frac{3}{2}tj$

EFFECTS OF WINDS AND CURRENTS

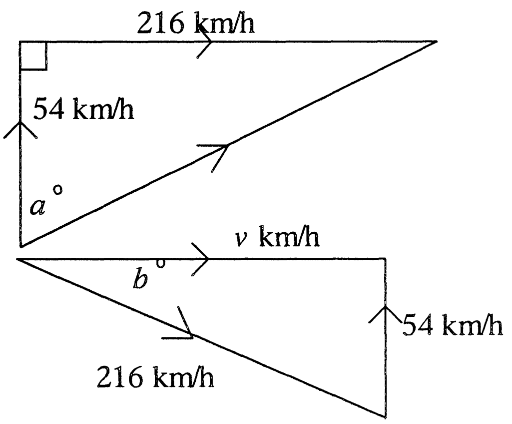
Worked Example

A wind is blowing from the north at 54 km/h. A plane can fly at 216 km/h.

a) If the pilot steers due east, on what bearing will the plane travel?

b) What course should the pilot set in order to fly due east? Calculate the actual speed of the plane.

Solution



Combine velocities of wind and plane using triangle law of vector addition.

tan $a° = 216/54 \Rightarrow a = 76$

sin $b° = 1/4 \Rightarrow b = 14{\cdot}5$

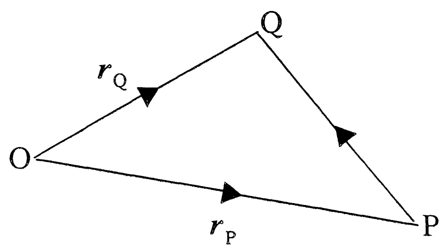
So set course on bearing of 104·5°

$v^2 = 216^2 - 45^2$

$\Rightarrow v = 209.1$

RELATIVE POSITION, VELOCITY AND ACCELERATION

The diagram on the right shows the positions of two bodies P and Q and a fixed origin O. With respect to O, P and Q have instantaneous position vectors r_P and r_Q. The instantaneous position vector of Q relative to P is $\overrightarrow{PQ}$.



Now $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$ and

So $\boxed{\overrightarrow{PQ} = r_Q - r_P}$

If P and Q are moving then the velocity of Q relative to P is $\dfrac{d(r_Q - r_P)}{dt}$

Now $\dfrac{d(r_Q - r_P)}{dt} = \dfrac{dr_Q}{dt} - \dfrac{dr_P}{dt} = v_Q - v_P$ and so

the velocity of Q relative to P is $\boxed{v_Q - v_P}$

Similarly, **the acceleration of Q relative to P** is $\boxed{a_Q - a_P}$

Worked Example

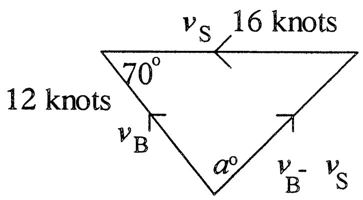
A ship is sailing due west at 16 knots and a boat is sailing at 12 knots on a bearing of 340°. Find the velocity of the boat relative to the ship.

Solution

If v_S is the velocity of the ship and v_B is the velocity of the boat then the velocity of the boat relative to the ship is $v_B - v_S$.

Ways of evaluating $v_B - v_S$ include *using trigonometry* and *resolving the velocities into components* and both methods are illustrated below.

<u>Method 1</u> (Using Trigonometry)



Cosine Rule: $|v_B - v_S|^2 = 16^2 + 12^2 - 2\times16\times12\times\cos 70°$

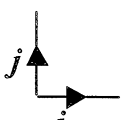
$\Rightarrow |v_B - v_S| = 16{\cdot}4$ knots

Sine Rule: $\dfrac{16}{\sin a°} = \dfrac{16\cdot4}{\sin 70°} \Rightarrow \sin a° = \dfrac{16\sin 70°}{16\cdot4}$

$\Rightarrow a = 66{\cdot}5$

So the velocity of the boat relative to the ship is 16·4 knots on a bearing of 046·5°

<u>Method 2</u> (By resolving velocities into components)



$v_S = -16i$ and $v_B = -12\sin20°\, i + 12\cos20°\, j = -4{\cdot}1i + 11{\cdot}3j$

So $v_B - v_S = -4{\cdot}1i + 11{\cdot}3j - (-16i) = 11.9\,i + 11{\cdot}3j$

$|v_B - v_S| = \sqrt{11\cdot9^2 + 11\cdot3^2} = 16{\cdot}4$ knots

angle with positive direction of x-axis $= \tan^{-1}\dfrac{11\cdot3}{11\cdot9} = 43{\cdot}5°$

Therefore bearing is 046·5°

COLLISION COURSES

In this class of problem we have two planes/ships/bodies moving with constant velocities and given their initial positions we have to show that they will, if they continue to move with these velocities, collide. There are basically two ways of tackling problems of this type and both are illustrated in the following worked example

Worked Example

At noon two ships P and Q are 10 kilometres apart with Q due south of P. P is sailing south east at a constant speed of 10 kmh^{-1} and Q is sailing due east at $5\sqrt{2}$ kmh^{-1}. Show that, if neither ship changes its velocity, they will collide and find the time, to the nearest minute, when the collision occurs.

Solution

Method 1

If $\boldsymbol{v}_P$ is the velocity of P and $\boldsymbol{v}_Q$ is the velocity of Q then the velocity of P relative to Q is $\boldsymbol{v}_P - \boldsymbol{v}_Q$. Q is initially due south of P and to show that the ships collide we need to show that the direction of $\boldsymbol{v}_P - \boldsymbol{v}_Q$ is due south.

Taking $\boldsymbol{i}$ and $\boldsymbol{j}$ to have directions east and north respectively then

$$\boldsymbol{v}_P = 10\cos 45^\circ \boldsymbol{i} - 10\sin 45^\circ \boldsymbol{j} = 5\sqrt{2}\boldsymbol{i} - 5\sqrt{2}\boldsymbol{j} \text{ and } \boldsymbol{v}_Q = 5\sqrt{2}\boldsymbol{i}$$

Hence $\boldsymbol{v}_P - \boldsymbol{v}_Q = -5\sqrt{2}\boldsymbol{i}$ which is due south and so the ships will collide.

Time to collision is $\dfrac{10}{5\sqrt{2}} = \sqrt{2}$ hours, and so ships will collide at 1.25 pm

Method 2

If $\boldsymbol{r}_P$ and $\boldsymbol{r}_Q$ are the position vectors of P and Q respectively then P and Q will collide if $\boldsymbol{r}_P = \boldsymbol{r}_Q$ at some instant.

Now $\boldsymbol{v}_P = 10\cos 45^\circ \boldsymbol{i} - 10\sin 45^\circ \boldsymbol{j} = 5\sqrt{2}\boldsymbol{i} - 5\sqrt{2}\boldsymbol{j} \Rightarrow \boldsymbol{r}_P = 5\sqrt{2}t\boldsymbol{i} - 5\sqrt{2}t\boldsymbol{j} + \boldsymbol{c}$

Taking the position of P at noon to be the origin, then $\boldsymbol{r}_P = \boldsymbol{0}$ when $t = 0$ so $\boldsymbol{c} = \boldsymbol{0}$ and hence $\boldsymbol{r}_P = 5\sqrt{2}t\boldsymbol{i} - 5\sqrt{2}t\boldsymbol{j}$

$\boldsymbol{v}_Q = 5\sqrt{2}\boldsymbol{i} \Rightarrow \boldsymbol{r}_Q = 5\sqrt{2}t\boldsymbol{i} + \boldsymbol{c}$ Now $\boldsymbol{r}_Q = -10\boldsymbol{j}$ when t = 0 so $\boldsymbol{c} = -10\boldsymbol{j}$ and hence $\boldsymbol{r}_Q = 5\sqrt{2}t\boldsymbol{i} - 10\boldsymbol{j}$

Consider $\boldsymbol{r}_P - \boldsymbol{r}_Q$:

$\boldsymbol{r}_P - \boldsymbol{r}_Q = 5\sqrt{2}t\boldsymbol{i} - 5\sqrt{2}t\boldsymbol{j} - (5\sqrt{2}t\boldsymbol{i} - 10\boldsymbol{j}) = -(5\sqrt{2}t - 10)\boldsymbol{j}$

Now $\boldsymbol{r}_P - \boldsymbol{r}_Q = \boldsymbol{0}$ when $5\sqrt{2}t - 10 = 0$ ie when t = $\sqrt{2}$

Thus ships will collide at 1.25 pm.

NEAREST APPROACH

In this class of problem two ships/planes/bodies are moving with constant velocities and we have to determine when they will be closest to one another. There are three ways in which we can tackle such problems and these are illustrated in the solution to the following worked example

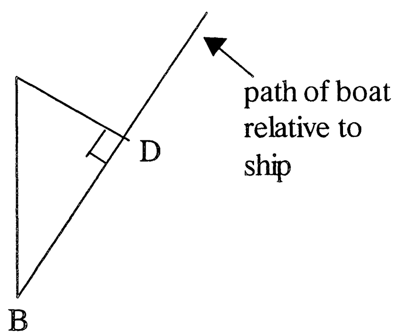
Worked Example

At noon a ship is 10 nautical miles due north of a boat. The ship is sailing due west at 16 knots and the boat is sailing at 12 knots on a bearing of 340°. Find the time when they are closest together and the distance apart at this time.

Solution

Method 1

To an observer on the ship the ship appears to be stationary and the boat is travelling with velocity $v_B - v_S$



Path of boat is in the direction of $v_B - v_S$.
We found previously that $v_B - v_S$ had magnitude 16·4 knots and direction on a bearing of 046·5°.

Closest distance = SD = BSsin46·5° = 7·25 nautical miles.

Time taken = $\dfrac{BD}{16.4} = \dfrac{10\cos 46\cdot 5^0}{16.4} = 0\cdot 420$h

Thus closest at 12.25pm

Method 2

Here we consider the position vectors of the ship and boat. In an earlier example we found that

$v_S = -16i$ and $v_B = -4\cdot 1i + 11\cdot 3j$

$v_S = -16i \Rightarrow r_S = -16ti + c$

Taking the position of the boat at noon as the origin $r_S = 10j$ when t = 0 so $c = 10j$

Hence $r_S = -16ti + 10j$

$v_B = -4\cdot 1i + 11\cdot 3j \Rightarrow r_B = -4\cdot 1ti + 11\cdot 3tj + c$

$r_B = \mathbf{0}$ when t = 0 thus $c = \mathbf{0}$ hence $r_B = -4\cdot 1ti + 11\cdot 3tj$

The position vector of the boat relative to the ship $= r_B - r_S$

Now $r_B - r_S = (-4{\cdot}1t + 16t)i + (11{\cdot}3t - 10)j$

$$|r_B - r_S|^2 = (11{\cdot}9t)^2 + (11{\cdot}3t - 10)^2 = 141{\cdot}6t^2 + 127{\cdot}7\,t^2 - 226t + 100$$
$$= 269{\cdot}3\,t^2 - 226t + 100$$

$$\frac{d}{dt}|r_B - r_S|^2 = 538{\cdot}6t - 266$$

Note it is more convenient to determine when $|r_B - r_S|^2$ has a stationary value rather than when $|r_B - r_S|$ has a stationary value.

Thus $|r_B - r_S|^2$ has a stationary value when $t = \dfrac{266}{538\cdot 6} = 0{\cdot}42$ h i.e. after 25 mins

This is clearly a minimum value and the minimum value is 52·58. Hence the closest distance apart is $\sqrt{52\cdot 58} = 7{\cdot}25$ nautical miles

Method 3

This depends on the fact that when S and B are closest together $r_B - r_S$ is perpendicular to $v_B - v_S$ and **is only appropriate for students who are familiar with and fluent in the use of the scalar product** of two vectors.

$(r_B - r_S) \perp r\ (v_B - v_S)$

$$\Rightarrow (r_B - r_S).(v_B - v_S) = 0$$
$$\Rightarrow [11{\cdot}9ti + (11{\cdot}3t - 10)j].(11.9i + 11{\cdot}3j) = 0$$
$$\Rightarrow 11{\cdot}9^2t + 11{\cdot}3(11{\cdot}3t - 10)$$
$$\Rightarrow 141{\cdot}6\,t + 127{\cdot}7t - 113 = 0$$
$$\Rightarrow t = \frac{113}{269\cdot 3} = 0{\cdot}42 \text{ h} = 25 \text{ min}$$

The closest distance can then be found by substituting 0·42 for t in $|r_B - r_S|$

RESOURCES/EXAMPLES

S&T Chapter 1 (Vectors)
Pages 9 –12; Chapter 10 Page 210
Chapter 2 (Distance, Velocity and Acceleration)
Pages 21 – 23; Chapter 10 Pages 219 – 229
Chapter 16 (Use of Calculus)
Pages 393-399; Ex 16.4, N^{os} 15 – 20, 33 – 38
Chapter 10 (Resultant Velocity and Relative Velocity)
Pages 223 – 247;Ex 10A (omit N^{o} 2) EXs10B-G, 10H(**A/B**)

RCS Chapter 4 (Kinematics in one dimension)
Pages 59, 60, 64 – 66 (using forces)
Chapter 2 (Kinematics)
Pages 38, 39: Ex 2J, Page 40, Ex 2K, N^{os} 1 – 6
Chapter 15 (Relative Motion)
Page 269 Ex15A;Pages 286 – 294, Ex 15B, 15C, 15D

TG Chapter 3 (Vectors and Forces)
Pages 29 – 31 (using forces);
Chapter 5 (Motion and Vectors)
Pages 62 - 66 Ex5.1B; Ex 5.2A N^{os} 1, 6, 7
Chapter 21 (Relative Motion)
Pages 368 – 374;Ex 21.1A N^{o} 1 – 10;Ex 21.1B, N^{o} 1–5, 7-10
Page 374 Consolidation Ex (**A/B**)
Chapter 7 (Motion with Variable Forces and Acceleration)
Pages 95 – 98; Ex 7.1A, N^{os} 1,2,5,8;
Pages 99,100; Ex 7.1B, N^{os} 1,5,6,7,8,10
Page 103;Ex 7.2A, N^{os} 1,5; Pages 104,105; Ex 7.2B, N^{os} 1,5,6

B&C Chapter 2 (Vectors. Components and Resultants….)
Pages 19 – 23; Chapter 13 Pages 420 – 422
Chapter 13 (Resultant Motion. Relative Motion)
Pages 420 – 431; Ex 13a; Page 424, Ex 13b N^{os} 1,2; Ex 13c
Pages 440 – 444, Ex 13e; Pages 448,449 Ex 13(**A/B**)
Chapter 4 (Velocity and Acceleration)
Pages 134,136

OG Chapter 6 (Vectors,……)
Page 292, Page 295, questions 6, 18
Chapter 7 (Kinematics of a Particle)
Pages 394 – 396
Pages 386-390; Ex 7.2:1, N^{os} 1,2,3,13,21,22,26
Pages 391,393; Ex 7.2:2, N^{os} 1 – 6,8,9,10,19,21
Pages 398 - 400, Ex 7.2.3, N^{os} 2, 4–7, 10–40, 42–52, 53–64

MOTION OF PROJECTILES IN A VERTICAL PLANE

CONTENT
know the meaning of the terms projectiles, velocity and angle of projection, trajectory, time of flight, range and constant gravity
solve the vector equation $\ddot{\underline{r}} = -g\underline{j}$ to obtain $\underline{r}$ in terms of its horizontal and vertical components
obtain and solve the equations of motion $\ddot{x} = 0$ and $\ddot{y} = -g$, obtaining expressions for $\dot{x}$, $\dot{y}$, x and y in any particular case
find the time of flight; greatest height reached and range of a projectile
find the maximum range of a projectile on a horizontal plane and the angle of projection to achieve this
find, and use, the equation of the trajectory of a projectile
solve problems in two-dimensional motion involving projectiles under constant gravitational force and neglecting air resistance.

Teaching notes

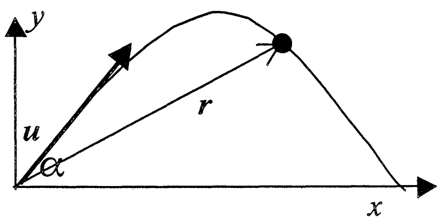
Projectiles are bodies that are thrown, dropped or launched into the air under the influence of gravity. Computer software packages and graphics calculators can be used to simulate the motion of a projectile.

In modelling the motion of a projectile certain assumptions are made, viz., that the motion is under **constant gravity** and that air resistance can be ignored. Furthermore, we consider the body to be a particle and so ignore any aerodynamic effects due to the shape of the body. Gravity acts vertically, if the unit vector j is taken to have direction vertically upwards then the motion of the projected particle is described by the vector equation $\ddot{r} = -gj$.

Some textbooks do not solve the vector equation $\ddot{r} = -gj$ but deal from the start with the equations for horizontal and vertical motion separately.

Solving the vector equation $\ddot{r} = -gj$

A projectile is launched from the origin, initial velocity u, angle of projection α



$\ddot{r} = -gj \Rightarrow \quad \dot{r} = -gtj + c_1$

when $t = 0$, $\dot{r} = u\cos\alpha i + u\sin\alpha j$

hence $\dot{r} = u\cos\alpha i + (u\sin\alpha - gt)j \quad (1)$

$(1) \Rightarrow r = ut\cos\alpha i + (ut\sin\alpha - \frac{1}{2}gt^2)j + c_2$

Initial position is O so $\boldsymbol{c}_2 = \boldsymbol{0}$ hence $\boldsymbol{r} = ut\cos\alpha\boldsymbol{i} + (ut\sin\alpha - \frac{1}{2}gt^2)\boldsymbol{j}$ (2)

Comparing (1) with $\dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$ we have $\dot{x} = u\cos\alpha$ and $\dot{y} = u\sin\alpha - gt$

Comparing (2) with $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ we have $x = ut\cos\alpha$ and $y = ut\sin\alpha - \frac{1}{2}gt^2$

Time of flight

$y = ut\sin\alpha - \frac{1}{2}gt^2$ and the projectile meets the plane when $y = 0$

i.e. when $ut\sin\alpha - \frac{1}{2}gt^2 = 0$ i.e. when $t = 0$ or $t = \dfrac{2u\sin\alpha}{g}$

$t = 0$ relates to the starting time and so $\boxed{\dfrac{2u\sin\alpha}{g}}$ is the time of the flight.

Range of projectile

The time of the flight is $\dfrac{2u\sin\alpha}{g}$ and we have found that $x = ut\cos\alpha$.

When $t = \dfrac{2u\sin\alpha}{g}$, $x = \dfrac{2u^2\sin\alpha\cos\alpha}{g} = \dfrac{u^2\sin 2\alpha}{g}$

and so the range of the projectile on the horizontal plane is $\mathbf{R} = \dfrac{u^2\sin 2\alpha}{g}$.

Note that when $t = \dfrac{2u\sin\alpha}{g}$, $\dot{y} = u\sin\alpha - g\dfrac{2u\sin\alpha}{g} = -u\sin\alpha$ and so the projectile hits the ground with a speed equal to its initial speed.

Maximum range on the horizontal plane

From the formulae for the range of the projectile it follows that for a given value of u the maximum range will occur when $\sin 2\alpha = 1$ i.e. when $\alpha = \frac{\pi}{4}$. Thus the maximum range on the horizontal plane is given by $\mathbf{R}_{\mathbf{max}} = \dfrac{u^2}{g}$, when $\alpha = \dfrac{\pi}{4}$

Greatest height

$y = ut\sin\alpha - \frac{1}{2}gt^2 \Rightarrow \dfrac{dy}{dt} = u\sin\alpha - gt$

y clearly has a maximum value and this will occur when $\dfrac{dy}{dt} = 0$ ie when $t = \dfrac{u\sin\alpha}{g}$

The maximum height is given by substituting this value of t in the equation for y.

The maximum height is $\dfrac{u^2\sin^2\alpha}{g}-\dfrac{1}{2}\dfrac{gu^2\sin^2\alpha}{g^2}$ i.e. $\mathbf{H}=\dfrac{u^2\sin^2\alpha}{g}$

Note, time to reach greatest height $=\dfrac{u\sin\alpha}{g}=\dfrac{1}{2}\dfrac{2u\sin\alpha}{g}=\dfrac{1}{2}$ time of whole flight.

Shape of the trajectory

We have found that $x=ut\cos\alpha$ - (a) and that $y=ut\sin\alpha-\frac{1}{2}gt^2$ - (b)

Now (a) and (b) are the parametric equations of a curve, with parameter t

From (a) we see that $t=\dfrac{x}{u\cos\alpha}$. Substituting this in (b) gives

$$y=u\sin\alpha\frac{x}{u\cos\alpha}-\frac{g}{2}\frac{x^2}{u^2\cos^2\alpha}\Rightarrow y=x\tan\alpha-\frac{g}{2u^2\cos^2\alpha}x^2 \quad \text{- (c)}$$

This is the Cartesian equation of the path of the projectile. As it is of the form $y=ax^2+bx$ it can be seen to be the equation of a **parabola**.

The equation $y=x\tan\alpha-\dfrac{g}{2u^2\cos^2\alpha}x^2$ provides another means of finding an expression for the range of the projectile on the horizontal plane. The projectile meets the horizontal plane through O when $y=0$ i.e. when $x\tan\alpha-\dfrac{g}{2u^2\cos^2\alpha}x^2=0$

ie when $x=0$ or when x = $\dfrac{2u^2\cos^2\alpha\tan\alpha}{g}=\dfrac{u^2\sin 2\alpha}{g}$. $x=0$ relates to the starting position and so the range on the horizontal plane is given by $R=\dfrac{u^2\sin 2\alpha}{g}$

The equation $y=x\tan\alpha-\dfrac{g}{2u^2\cos^2\alpha}x^2$ can be written as $y=x\tan\alpha-\dfrac{g\sec^2\alpha}{2u^2}x^2$

and since $\sec^2\alpha=1+\tan^2\alpha$ this becomes $y=x\tan\alpha-\dfrac{g}{2u^2}(1+\tan^2\alpha)x^2$

In this form, if we are given values for x, y and u, we have a quadratic equation in $\tan\alpha$. The two solutions for α, $0<\alpha<\frac{\pi}{2}$, give the two angles of projection of the projectile with initial speed u required for the projectile to pass through the point (x,y)

SOLVING PROBLEMS INVOLVING PROJECTILES

There are three possible approaches to tackling problems involving projectiles.

i.) From first principles, starting from either $\ddot{r} = -gj$ or $\ddot{x} = 0$ and $\ddot{y} = -g$

ii.) Working with the horizontal and vertical components of the motion separately, using speed/distance/time for the horizontal direction and Newton's equations of motion with constant acceleration in the vertical direction.

iii.) For appropriate parts, quoting and using the formulae established in this topic, eg Range, Greatest Height, etc.

All these approaches are acceptable, with i) being the more favoured approach.

WORKED EXAMPLES

Example 1

A golf ball is struck with initial velocity $25\boldsymbol{i} + 15\boldsymbol{j}$, measured in ms^{-1}, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are horizontal and vertical unit vectors. Find

a) its velocity $\boldsymbol{v}$ after t seconds
b) its position vector $\boldsymbol{r}$ at this instant
c) its time of flight before pitching on to a horizontal fairway
d) its horizontal range
e) its maximum height (assume that $g = 10\text{ms}^{-2}$)

Solution

Method 1

a) The acceleration $\boldsymbol{a}$ of the golf ball at any instant is $-g\boldsymbol{j}$.

Hence its velocity, t seconds after being hit, is $\boldsymbol{v}$ is given by $\frac{d\boldsymbol{v}}{dt} = -g\boldsymbol{j}$

so $\boldsymbol{v} = -gt\boldsymbol{j} + \boldsymbol{c}_1$ Now $\boldsymbol{v} = 25\boldsymbol{i} + 15\boldsymbol{j}$ when $t = 0$, so $\boldsymbol{c}_1 = 25\boldsymbol{i} + 15\boldsymbol{j}$

Thus $\boldsymbol{v} = 25\boldsymbol{i} + (15 - gt)\boldsymbol{j}$

b) $\boldsymbol{v} = 25\boldsymbol{i} + (15 - gt)\boldsymbol{j} \Rightarrow \boldsymbol{r} = 25t\boldsymbol{i} + (15t - \frac{1}{2}gt^2)\boldsymbol{j} + \boldsymbol{c}_2$

Now $\boldsymbol{r} = \boldsymbol{0}$ when $t = 0$, so $\boldsymbol{c}_2 = \boldsymbol{0}$ and thus $\boldsymbol{r} = 25t\boldsymbol{i} + (15t - \frac{1}{2}gt^2)\boldsymbol{j}$

c) The golf ball strikes the fairway when $\boldsymbol{r} = R\boldsymbol{i}$ ie when $15t - \frac{1}{2}gt^2 = 0$

$15t - \frac{1}{2}gt^2 = 0 \Rightarrow \frac{1}{2}(30 - gt)t = 0 \Rightarrow t = 0$ or $t = 3$

Hence golf ball strikes fairway after 3 seconds.

d) When $t = 3$, $\boldsymbol{r} = 75\boldsymbol{i}$ and so horizontal range is 75 metres.

e) $r = 25ti + (15t - \frac{1}{2}gt^2)j$ and so maximum height occurs when $15t - \frac{1}{2}gt^2$ has a maximum value i.e. when $\frac{d}{dt}(15t - \frac{1}{2}gt^2) = 0$ i.e. when $15 - gt = 0$ ie $t = 1{\cdot}5$

f) Hence maximum height is $15\times1{\cdot}5 - \frac{1}{2}\times10\times1{\cdot}5^2 = 1{\cdot}5\,(15 - 5\times1{\cdot}5) = 11{\cdot}25$ metres

Method 2

a) Horizontal component of velocity is constant = 25 ms^{-1}

Vertical component of velocity given by "$v = u + at$" where $u = 15$ and $a = g = -10$

so vertical component is $15 - 10t$ hence $v = 25i + (15 - 10t)j$

b) Distance travelled horizontally in time t is given by "$s = ut + \frac{1}{2}at^2$" where $a = 0$
so distance travelled horizontally in t seconds is $25t$ metres
Distance travelled vertically in t seconds given by "$s = ut + \frac{1}{2}at^2$" where
$a = g = -10$, so vertical distance is $15t - 5t^2$ metres. Hence $r = 25ti + 5(3t - t^2)j$

c) Vertical distance = 0 when $15t - 5t^2 = 0$ i.e. when $t = 0$ or $t = 3$

d) Distance travelled horizontally in 3 seconds is 75 metres
ie horizontal range is 75 metres

e) Height is maximum when vertical component of velocity = 0 $\Rightarrow 15 - 10t = 0$
$\Rightarrow t = 1{\cdot}5 \Rightarrow H = 15\times1{\cdot}5 - 5\times1{\cdot}5^2 = 11{\cdot}25$ metres

Example 2
Suppose that in the previous example the golfer is hitting the ball to the centre of a horizontal green, level with her feet, and 75 metres distant. Between her and the centre of the green 60 metres away is a tree of height 7 metres. Will the ball clear the tree?

Solution
We found that $r = 25ti + (15t - \frac{1}{2}gt^2)j = 25ti + (15t - 5t^2)j$
The ball has travelled 60 metres horizontally when $25t = 60$ i.e. when $t = 2{\cdot}4$
When $t = 2{\cdot}4$ the vertical height is $5\times2{\cdot}4(3 - 2{\cdot}4)$ metres i.e. $7{\cdot}2$ metres
Thus the ball will clear the tree.

Example 3
It is claimed that the record hit at cricket was 160 metres. Find the minimum initial speed of the ball (Assume g = 10ms^{-1}).

Solution

The horizontal range of a projectile is given by, "R = $\frac{u^2 \sin 2\alpha}{g}$".

Hence $\frac{u^2 \sin 2\alpha}{g}$ = 160 and so $u^2 = \frac{160g}{\sin 2\alpha}$

The required minimum initial velocity will occur when $\alpha = \frac{\pi}{4}$.

$u = \sqrt{160g} = \sqrt{1600} = 40$

Thus minimum initial velocity required is 40 ms^{-1} with an angle of projection of $\frac{\pi}{4}$.

RESOURCES/EXAMPLES

S&T Chapter 12 (Projectiles)

Pages 277 – 290; Exs 12A, 12B
Pages 291 – 293 ; Ex 12C N^{os} 1 – 12
Pages 299 – 304; Ex 12E (**all at A/B**)

RCS Chapter 9 (Projectiles)

Pages 168 – 173; Exs 9A, 9B
Pages 174 – 176; Ex 9C
Pages 177 – 180; Ex 9D, 9E
Pages 181 – 183; Ex Examination Questions (**A/B**)

TG Chapter 8 (Projectiles)

Pages 109 – 122; Ex 8.1A, Ex 8.1B, 8.2A (omit 7,8) 8.2B (omit 7,8)
Pages 126 – 128; Consolidation Exercise (**mostly A/B**)

B&C Chapter 9 (Projectiles)

Pages 267 – 270; Ex 9a: Pages 271 – 278; Ex 9b
Pages 279 – 286; Ex 9c, N^{os} 1 – 14
Pages 288 – 292; Ex 9 (omit 13, 16, 19) (**mostly A/B**)

OG Chapter 7 (Kinematics of a Particle)

Pages 401 – 403; Ex 7.3:1; Page 406; Ex 7.3:2 N^{os} 1 – 18
Pages 407, 408; Ex 7.3:2, N^{os} 19 – 36
Pages 410, 411; Ex 7.3:2, N^{os} 37 – 66

FORCE AND NEWTON'S LAWS OF MOTION

CONTENT
understand the terms mass, force, weight, momentum, balanced and unbalanced forces, resultant force, equilibrium, resistive forces know Newton's first and third laws of motion resolve forces in two dimensions to find their components combine forces to find resultant force understand the concept of static and dynamic friction and limiting friction understand the terms frictional force, normal reaction, coefficient of friction μ, angle of friction λ and know the equations $F = \mu R$ and $\mu = \tan\lambda$ solve problems involving a particle or body in equilibrium under the action of certain forces solve problems involving friction and problems on inclined planes

Teaching notes

Equilibrium

If a particle is at rest or moving with constant velocity it is said to be in **equilibrium**. By **Newton's First Law of Motion**, a body is in equilibrium if the net force acting on it is zero, i.e. the system of forces acting on the body is **balanced**.

A typical problem in this section of the Unit involves finding the magnitude of some force within a system of balanced forces acting on a body. In their earlier studies in other subjects, students may have tackled such problems by using, either the *Triangle of Forces* to show that the resultant force was zero or *Lami's Theorem* or both. Some textbooks have a section devoted to these procedures.

The method recommended in this unit, however, is to:

i) resolve the forces in two perpendicular directions

ii) show that the components in each of these directions 'balance out'

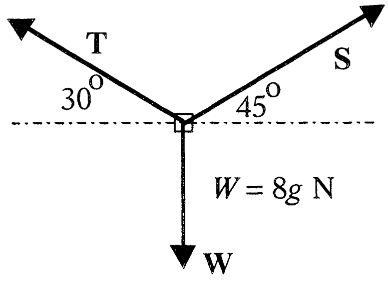
This method applies to any number of forces acting on a body.

In the assessment of this unit solutions by scale drawing are not acceptable.

For information, the following example is tackled by using the three different methods.

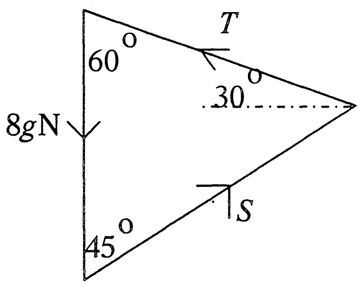
Example

A mass of 8kg hangs in equilibrium, suspended by two light, inelastic strings making angles of 30° and 45° with the horizontal, as shown in the diagram on the right. Calculate the tensions in the two strings.



Solution

Method 1 (using triangle of forces – this applies only to **3** forces)



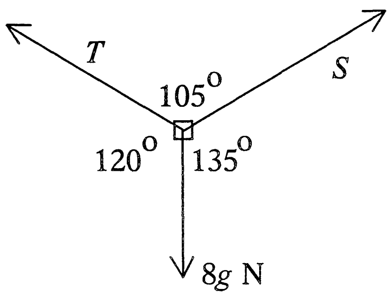
Using the Sine Rule in the triangle

$$\frac{T}{\sin 45^\circ} = \frac{S}{\sin 60^\circ} = \frac{8g}{\sin 75^\circ}$$

Giving $T = \dfrac{8g \sin 45^\circ}{\sin 75^\circ} = 57 \cdot 4\text{N}$

and $S = \dfrac{8g \sin 60^\circ}{\sin 75^\circ} = 70 \cdot 3\text{N}$

Method 2 (using Lami's Theorem – applies only to **3** forces and is **not** in syllabus)



Lami's Theorem gives

$$\frac{T}{\sin 135^\circ} = \frac{S}{\sin 120^\circ} = \frac{8g}{\sin 105^\circ}$$

Hence $T = \dfrac{8g \sin 135^\circ}{\sin 105^\circ} = 57 \cdot 4\text{N}$

and $S = \dfrac{8g \sin 120^\circ}{\sin 105^\circ} = 70 \cdot 3\text{N}$

Method 3 **(recommended)**

Body in equilibrium:

Resolving horizontally: $T\cos 30^\circ = S\cos 45^\circ$ (1)

Resolving vertically: $T\sin 30^\circ + S\sin 45^\circ = 8g$ (2)

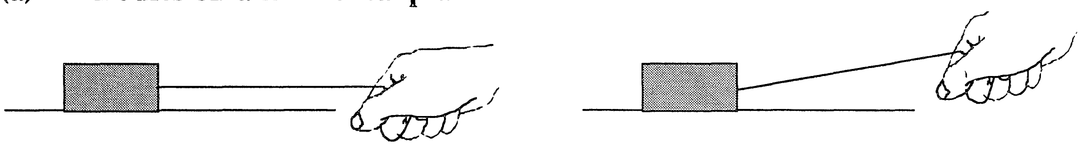
From (1) $S = \dfrac{T\cos 30^\circ}{\cos 45^\circ}$ substituting for S in (2) $T\sin 30^\circ + T\cos 30^\circ \tan 45^\circ = 8g$

Thus $T = \dfrac{8g}{\sin 30^\circ + \cos 30^\circ} = 57 \cdot 4\text{N}$

and $S = \dfrac{57 \cdot 4 \cos 30^\circ}{\cos 45^\circ} = 70 \cdot 3\text{N}$

PROBLEMS INVOLVING BODIES AT REST UNDER THE ACTION OF SEVERAL FORCES

(a) Bodies on a horizontal plane



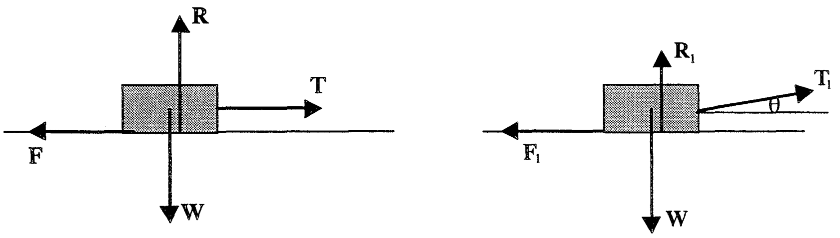
The above diagrams show a block at rest on a horizontal plane being "pulled" by a string which, on the left is parallel to the plane and on the right is inclined at an angle to the horizontal. Students should be aware of the forces acting on the block, namely

a. the **weight** of the block, **W**, acting vertically downwards

b. the **normal reaction**, **R**, of the plane acting at right angles to the plane (The block exerts a force on the plane and the plane exerts an equal and opposite force on the block – an illustration of **Newton's third law**)

c. the **tension** in the string, **T**, acting along the string

d. a **resistive force**, **F**, the **friction** between the block and the plane as shown in the diagrams below



The **weight** of the block, like the other forces is measured in newtons.
$W = mg$, where m is the mass of the block in kilograms and g is the acceleration due to gravity, measured in ms^{-2}.

Friction opposes motion. When there is no motion, as here, we have **static friction** and when there is motion we have kinetic or **dynamic friction** and, when the block is on the point of moving we have **limiting friction.**

Limiting Friction =μ(Normal Reaction)

μ is a constant, called the **coefficient of friction**, which depends on the materials of the block and plane. When $F < \mu R$ the block is stationary and when $F = \mu R$ the block is on the point of moving.

For dynamic friction $\frac{F}{R}$ is also constant, but less than for limiting friction.

In the situations described above, if there is no motion (or if the block moves with constant velocity) then by Newton's First Law the resultant force acting on the block is zero.

Suppose that in the above two situations the block is on the point of moving

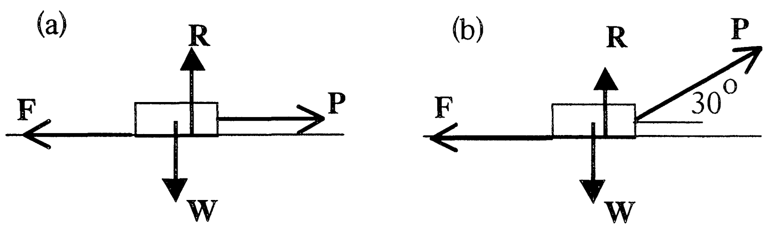
Horizontal String	Inclined String
No slipping: $F = \mu R$	No slipping: $F_1 = \mu R_1$
No motion:	No motion:
Resolving vertically	Resolving vertically
$R = W = mg$	$R_1 + T_1 \sin\theta = mg$
Resolving horizontally	Resolving horizontally
$T = F = \mu R$	$T_1 \cos\theta = F_1 = \mu R_1$

WORKED EXAMPLES

Example 1

A 5 kg block of wood is at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is 0·6.

Calculate the magnitude of the force **P** which is necessary for motion to occur if **P** is applied to the block (a) horizontally and (b) at an angle of 30° above the horizontal, as shown in the diagrams.



Solution

(a) $R = W = 5g$ and $P = F$ In limiting equilibrium $F = \mu R$
Hence $P = 0{\cdot}6 \times 5 \times 9{\cdot}8 = 29{\cdot}4$ N

(b) Resolving vertically $R + P\sin 30° = 5g$
Resolving horizontally $F = P\cos 30°$
In limiting equilibrium $F = \mu R$. Thus $P\cos 30° = 0{\cdot}6(5g - P\sin 30°)$

$$P(\frac{\sqrt{3}}{2} + 0{\cdot}3) = 29{\cdot}4 \Rightarrow P = 25{\cdot}2\text{N}$$

So in (a) the force must just exceed 29·4 N and in (b) 25·2 N for motion to take place.

Example 2

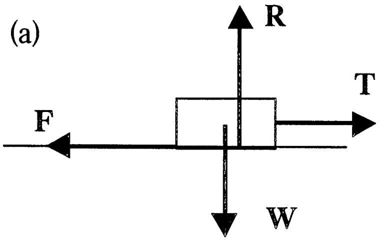
A block of mass 20 kilograms rests on a horizontal plane whose coefficient of friction is 0·4. Find the least force required to move the block if it acts:

a) horizontally; b) at 30° above the horizontal: c) at 30° below the horizontal

d) at the most favourable angle.

Solution

(a)



On the point of moving: $F = 0{\cdot}4R$

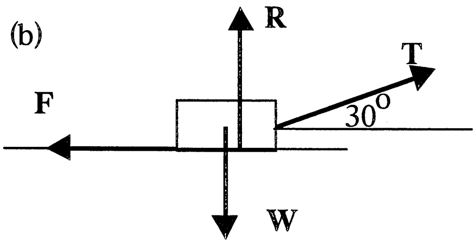
Resolving vertically:

$$R = W = 20g = 20 \times 9{\cdot}8 \text{ N} = 196 \text{ N}$$

Resolving horizontally:

$$T = F = 0{\cdot}4R = 0{\cdot}4 \times 196 \text{ N} = 78{\cdot}4 \text{ N}$$

(b)



On the point of moving: $F = 0{\cdot}4R$

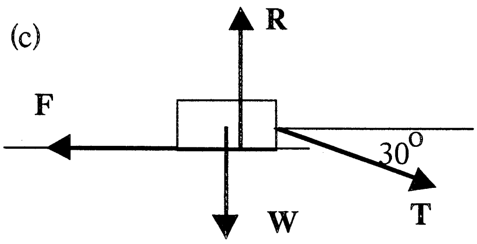
Resolving vertically: $R + T\sin 30° = W = 20g$

Resolving horizontally: $T\cos 30° = F = 0{\cdot}4R$

$\therefore\ T\cos 30° = 0{\cdot}4(20g - T\sin 30°)$

$$\therefore T = \frac{8g}{\cos 30° + 0{\cdot}4\sin 30°} = 73{\cdot}5 \text{ N}$$

(c)



On the point of moving: $F = 0{\cdot}4R$

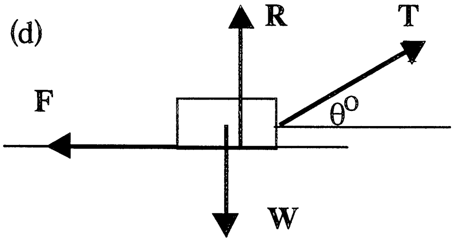
Resolving vertically: $R - T\sin 30° = W = 20g$

Resolving horizontally: $T\cos 30° = F = 0{\cdot}4R$

$\therefore\ T\cos 30° = 0{\cdot}4(20g + T\sin 30°)$

$$\therefore T = \frac{8g}{\cos 30° - 0{\cdot}4\sin 30°} = 117{\cdot}7 \text{ N}$$

(d)



On the point of moving: $F = 0{\cdot}4R$

Resolving vertically: $R + T\sin\theta° = W = 20g$

Resolving horizontally: $T\cos\theta° = F = 0{\cdot}4R$

$\therefore\ T\cos\theta° = 0{\cdot}4(20g - T\sin\theta°)$

$$\therefore T = \frac{8g}{\cos\theta° + 0{\cdot}4\sin\theta°} = \frac{8g}{\sqrt{1{\cdot}16}\cos(\theta - 21{\cdot}8)°} \text{ N}$$

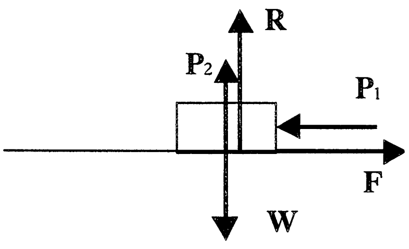
$$\therefore T \geq \frac{8g}{\sqrt{1{\cdot}16}} = 72.8 \text{ N}$$

Thus, the least force required to move the block if it acts (a) horizontally, is 78·4 N; (b) at 30° above the horizontal, is 73·5 N; (c) at 30° below the horizontal, is 117·7 N and (d) in the most favourable direction, 21·8° above horizontal, is 72·8 N

Note: Candidates would not be expected to recall how to express $a\cos\theta° + b\sin\theta°$ as $k\cos(\theta - \alpha)°$. Part (d) was included for 'completeness'.

Example 3
A worker tries unsuccessfully to move a 35 kg crate, resting on a horizontal floor, by pushing the crate horizontally with a force of 110 N. A second worker decides to help. What is the minimum pulling force she would need to apply (i) vertically (ii) horizontally to help move the crate if the coefficient of friction is 0·37?

Solution
(i) pulling vertically

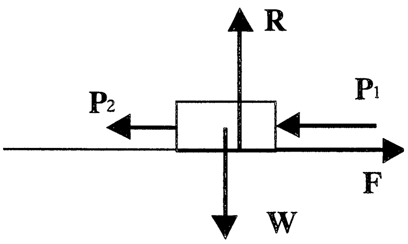


On the point of moving $F = \mu R$
Resolving vertically: $R + P_2 = W = 35g$

Resolving horizontally: $P_1 = F$ i.e $110 = 0{\cdot}37R$

$\therefore R = \dfrac{110}{0\cdot37} = 297\cdot3$
Hence $P_2 = 35g - 297{\cdot}3 = 45{\cdot}7$ N

(ii) pulling horizontally



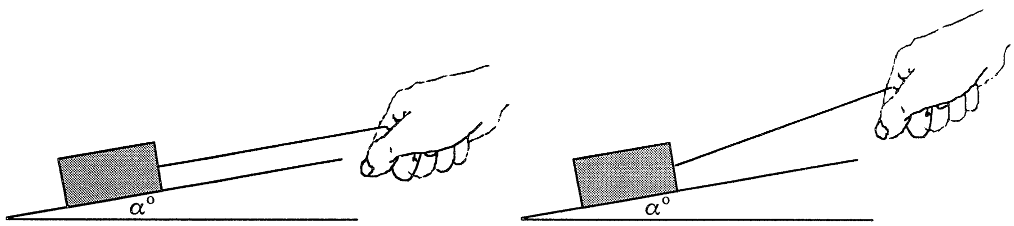
On the point of moving $F = \mu R$
Resolving vertically: $R = W = 35g$

Resolving horizontally: $P_2 + P_1 = F$

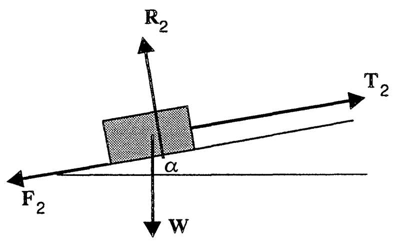
Thus $P_2 + 110 = 0{\cdot}37R = 126{\cdot}9$ $\therefore$ $P_2 = 16{\cdot}9$

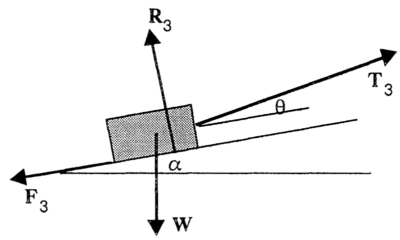
Minimum force of 16·9 N is required horizontally

(b) Bodies on an inclined plane



The above diagrams show a block at rest on a plane inclined at an angle α to the horizontal being "pulled" by a string which, on the left is parallel to the plane and on the right is inclined at an angle to the plane. Students should be aware that the forces acting on the block in each case are:

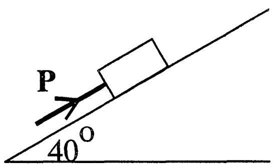




String parallel to inclined plane	String at an angle θ to inclined plane
On the point of slipping: $F_2 = \mu R_2$	On the point of slipping: $F_3 = \mu R_3$
No motion:	No motion:
Resolving perpendicular to the plane:	Resolving perpendicular to the plane:
$R_2 = W\cos\alpha = mg\cos\alpha$	$R_3 + T_3\sin\theta = W\cos\alpha = mg\cos\alpha$
Resolving parallel to the plane:	Resolving parallel to the plane:
$T_2 = F_2 = \mu R_2$	$T_3\cos\theta = F_3 = \mu R_3$

WORKED EXAMPLES

Example 1

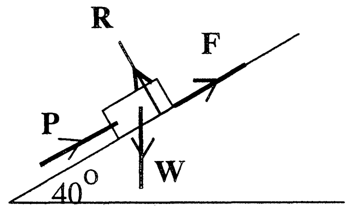


A block of wood of mass 6kg is placed on a rough slope, inclined at an angle of 40° to the horizontal. It is held in position by the action of a force **P** newtons acting parallel to the slope. The coefficient of friction between the block of wood and the surface of the slope is 0·4.

Find the greatest and least values of P for the block to remain stationary on the slope.

Solution

(i)



On the point of slipping down the slope

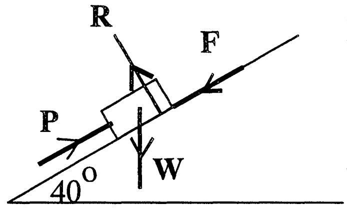
Friction acts up the slope and $F = \mu R = 0{\cdot}4R$

Resolving parallel to the slope: $P + F = 6g\sin 40°$

Resolving perpendicular to the slope: $R = 6g\cos 40°$

Thus $P + 0{\cdot}4 \times 6g\cos 40° = 6g\sin 40°$

$\Rightarrow P = g(6\sin 40° - 2{\cdot}4\cos 40°) = 19{\cdot}8$ N

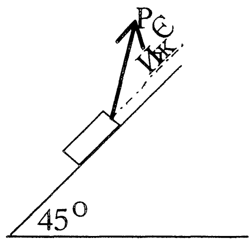
(ii) 

On the point of moving up the slope
Friction acts down the slope and $F = \mu R = 0{\cdot}4R$
Resolving parallel to the slope: $P = F + 6g\sin 40°$
Resolving perpendicular to the slope: $R = 6g\cos 40°$
Thus $P = 0{\cdot}4 \times 6g\cos 40° + 6g\sin 40°$
$\Rightarrow P = g(2{\cdot}4\cos 40° + 6\sin 40°) = 55{\cdot}8$ N

Thus $\quad 19{\cdot}8 \text{ N} \le P \le 55{\cdot}8 \text{ N}$

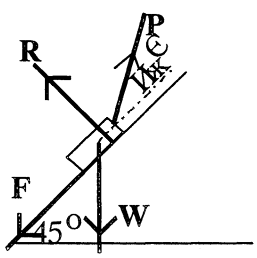
Example 2

A block of mass 4kg rests on a plane inclined at an angle of 45° to the horizontal, under the action of a force of magnitude **P** acting upwards at an angle of 30° to the line of greatest slope of the plane.



If the coefficient of friction is $\frac{1}{\sqrt{2}}$, calculate the magnitude of **P** if the particle is on the point of moving up the plane.

Solution



If the particle is on the point of moving up the plane then friction is acting down the plane and $F = \mu R$
Resolving parallel to the plane

$$P\cos 30° = F + W\sin 45° = F + 4g\sin 45°$$

Resolving perpendicular to the plane $R + P\sin 30° = W\cos 45°$

$$\Rightarrow R + P\sin 30° = 4g\cos 45° \quad \Rightarrow R = \frac{4g}{\sqrt{2}} - \frac{P}{2}$$

Hence $P\dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{2}}\left(\dfrac{4g}{\sqrt{2}} - \dfrac{P}{2}\right) + \dfrac{4g}{\sqrt{2}} = 2g - \dfrac{P}{2\sqrt{2}} + 2\sqrt{2}g$

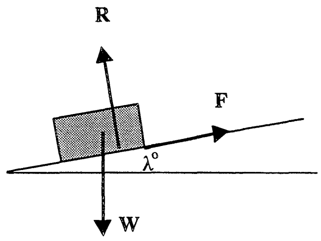
$$\Rightarrow \quad P\left(\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2}}\right) = (2 + 2\sqrt{2})g$$

$$\Rightarrow \quad P = \frac{(4\sqrt{2} + 8) \times 9{\cdot}8}{\sqrt{6} + 1} = 38{\cdot}8 \text{ N}$$

ANGLE OF FRICTION

A block of mass *m* kilograms rests on a plane, which is gradually tilted until the block is on the point of moving down the plane. Suppose the angle of the plane to the horizontal is λ when the block is on the point of slipping and the coefficient of friction between the block and the plane is μ.

Since the block is on the point of moving down the plane, friction is acting up the plane.



On the point of slipping: $F = \mu R$

No motion:

Resolving perpendicular to the plane

$$R = W\cos\lambda = mg\cos\lambda$$

Resolving parallel to the plane

$$F = mg\sin\lambda$$

Thus $\mu = \dfrac{F}{R} = \dfrac{mg\sin\lambda}{mg\cos\lambda} = \dfrac{\sin\lambda}{\cos\lambda} = \tan\lambda$

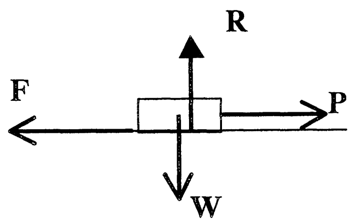
λ is known as the angle of friction.

WORKED EXAMPLES

Example 1

When a horizontal force **P** of magnitude 20 N is applied to a body of mass 6kg, resting on a rough surface, the body is in limiting equilibrium. Calculate the angle of friction.

Solution



$R = W = 6g$ and $F = P = 20$

In limiting equilibrium $F = \mu R$

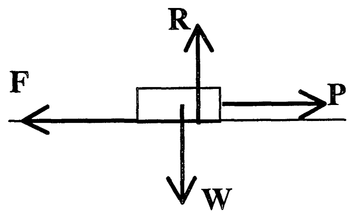
Hence $\mu \times 6g = 20 \Rightarrow \mu = 0{\cdot}340$

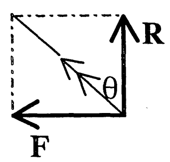
$\tan \lambda = \mu \Rightarrow \lambda = 18{\cdot}8°$

Example 2

A block of mass m kilograms, resting on a horizontal plane, is acted upon by a horizontal force **P** of magnitude P newtons. Show that if the resultant of the frictional force and the normal reaction makes an angle of θ with the vertical then $\tan\theta \le \mu$.

Solution





Since the block is at rest, $F \le \mu R$

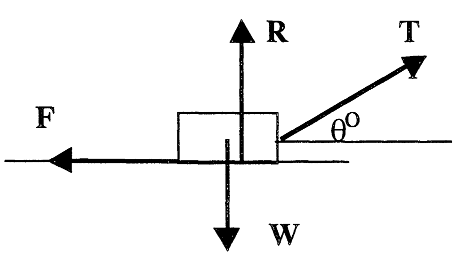
Now, $\tan\theta = \dfrac{F}{R} \le \mu$

Note: Since $\mu = \tan\lambda$ it follows that $\theta \le \lambda$. If $\theta < \lambda$, the block remains stationary and when $\theta = \lambda$ it begins to move. This is the basis of an alternative definition of the angle of friction.

Example 3

A block of mass m kilograms is lying on a horizontal plane and is acted upon by a force **T** of magnitude T newtons which makes an angle of θ above the horizontal. If the block is on the point of moving, find the value of θ such that T is a minimum.

Solution



As the block is on the point of moving $F = \mu R$

Resolving vertically: $R + T\sin\theta = W = mg$

Resolving horizontally: $T\cos\theta = F$

Thus $T\cos\theta = F = \mu R = \mu(mg - T\sin\theta)$

$$\therefore\ T = \frac{\mu mg}{\cos\theta + \mu\sin\theta}$$

Now, $\mu = \tan\lambda$, so $T = \dfrac{\mu mg}{\cos\theta + \mu\sin\theta} = \dfrac{\dfrac{\sin\lambda}{\cos\lambda}mg}{\cos\theta + \dfrac{\sin\lambda}{\cos\lambda}\sin\theta} = \dfrac{\mu mg\sin\lambda}{\cos\theta\cos\lambda + \sin\theta\sin\lambda}$

Thus $T = \dfrac{\mu mg\sin\lambda}{\cos(\theta - \lambda)}$ and so T has a minimum value of $\mu mg\sin\lambda$ when $\theta = \lambda$

RESOURCES/EXAMPLES

S&T

Chapter 3 (Force and Newton's Laws)
- Pages 41,49; Ex 3A Nos 1, 2, 13

Chapter 4 (Resultants and Components of Forces)
- Pages 64 – 79; Ex 4A, 4C (select) Ex 4B Nos 1 - 5

Chapter 5 (Equilibrium and Acceleration under Constant Forces)
- Pages 80,81,85 Ex 5A Nos 8,9,10,Pages 87-89,91;Ex5B,Nos1-9,10-18
- Pages 95 – 101 Ex 5C Nos 6 – 9, 14 – 18; Ex 5D Nos 4 – 6,13,14

Chapter 6 (Friction)
- Pages 107 – 118; Ex 6A, Nos 1,3,4,12,13 Ex 6B, Nos 1 – 16, 21, 22
- Pages 122 – 123; Ex 6C, Nos 1,2,3(a),(b),(c),4,5,11,12
- Page 128; Ex6D, Nos 1 - 3

RCS

Chapter 1 (Modelling)
- Pages 5 – 12 (background)

Chapter 3 (Force and Motion)
- Pages 44,45; Ex 3A; Page 53 (start)

Chapter 4 (Resolving)
- Pages 59, 60; Ex 4A No 1; Ex 4B Nos2-4,6, 8; Pages 64 – 66; Ex 4D
- Pages 67,68; Ex 4E, Nos 2,3,4,6,7,8 Page 70 Exam q 3

Chapter 5 (Friction)
- Pages 80–84; Ex 5A, Nos 1–8; Ex 5B, Nos 1,5,6,7,10-12 Ex 5C, Nos 4,5
- Pages 88 -91; Ex 5E, Nos 1 – 5, Page 92, exam question 1

TG

Chapter 2 (Force)
- Pages 8 – 11; Page 13; Ex 2.2A, Nos 1,2,5; Ex 2.2B Nos 1,2
- Pages 16,17; Ex2.3A, Nos 8,9,10,11,Ex 2.3B, Nos 10,11

Chapter 3 (Vectors and Forces)
- Pages 28 – 33 (ignore pulleys); Ex 3.1A, 3.1B
- Pages 34 – 38; Exam. 3.4, 3.7;Ex3.2A, Nos1–3,9, Ex3.2B, Nos 1-6,9

Chapter 6 (Newton's First and Second Laws)
- Pages 77, 78 Ex 6.1A, Nos 1,2,6,7,9,10; Ex 6.1B, Nos 7,8
- Page 82 Ex 6.2A, No 10; Ex6.2B, Nos 6,7,9

B&C

Chapter 2 (Vectors. Components and Resultants….)
- Pages 19,20; Ex2b, Nos 1,3

Chapter 3 (Coplaner Forces in Equilibrium. Friction)
- Pages 66 – 69; Ex 3a, Nos 3 – 6;
- Page 76, Ex3b; Page 78, Nos 1-4,7
- Pages 82-84; Ex 3C, Nos 1,2,3,5,6

Chapter 5 (Newton's Laws of Motion)
- Pages 149, 150, 157 Ex 5b, Nos4,7; Page176, Ex 5, Nos 3,6

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Chapter 8 (Forces on a Particle)
- Pages 420 – 424; Ex8.1:1; Page 427
- Pages 428,429; Ex 8.1:3, Nos 10,11,12,14 - 18
- Pages 430, 431; Page 437; Ex 8.2:1, Nos 1 – 4, 26 - 29
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CONTENT
know Newton's second law of motion; that force is the rate of change of momentum, and derive the equation $\underline{F} = m\underline{a}$
use this equation to form equations of motion to model practical problems on motion in a straight line
solve such equations modelling motion in one dimension, including cases where the acceleration is dependent on time.

Teaching notes

Deriving the equation $\mathbf{F} = ma$

Newton's Second Law of Motion states that the rate of change of momentum of a body is directly proportional to the applied force and is in the direction of that force. The **momentum** of a body of mass m kilograms and velocity v is the vector quantity mv. If it is acted upon by a force **F** then:

Newton's Second Law $\Leftrightarrow \mathbf{F} = k\frac{d(mv)}{dt}$, where k is a constant

If the mass remains constant then $\frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$. Thus $\mathbf{F} = kma$

The unit for force, the newton, is chosen such that a force of 1 newton acting on a mass of 1 kilogram will cause an acceleration of 1 ms^{-2}, so $k = 1$ and hence $\mathbf{F} = ma$

WORKED EXAMPLES

Example 1

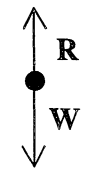
A pile of mass 6000kg starts to penetrate the ground with a speed of 7ms^{-1} and penetrates the ground to a depth of 25 cm before being brought to rest. Calculate the resistance of the ground to penetration, assuming this resistance is constant.

Solution

Using $v^2 = u^2 + 2as$

$0 = 49 + 2a \times 0{\cdot}25$

$\Rightarrow \quad a = -\frac{49}{0{\cdot}5} = -98\text{ms}^{-2}$



Equation of motion gives

$W - R = ma$

$6000g - R = -6000 \times 98$

$\Rightarrow \quad R = 646800 \text{ N}$

Example 2

A body of mass 4 kg moves along a horizontal straight line under the action of two forces in the direction of the line. One force is a variable, propulsive force of magnitude F newtons and the other is a constant, resistive force of magnitude 12 newtons. The body starts from rest and acquires a speed of v ms^{-1}, in time t seconds, given by $v = 10t - t^2$ for $0 \le t \le 6$.

a) Find an expression for F in terms of t.

b) Calculate the value of F when the body has reached maximum speed.

Solution

a) Using the equation of motion

$$F - 12 = 4a$$
$$\Rightarrow F = 4a + 12$$

but $v = 10t - t^2$ $\quad \Rightarrow a = \dfrac{dv}{dt} = 10 - 2t$

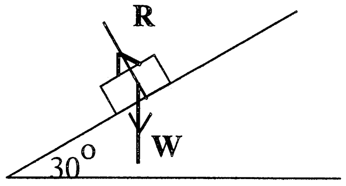
$$\Rightarrow F = 4(10 - 2t) + 12 = 52 - 8t$$

b) Max speed is when acceleration = 0: $a = 0 \Rightarrow 10 - 2t = 0 \Rightarrow t = 5$
When $t = 5$, $F = 52 - 40 = 12$ N

Example 3

A block is projected with speed 7ms^{-1} up a smooth plane which is inclined at 30° to the horizontal. Find the distance it travels before coming to rest momentarily.

Solution



Resolving parallel to the plane (in direction of motion)

Resultant force has magnitude

$-W\sin 30° = -mg \sin 30°$

Thus equation of motion is

$-mg \sin 30° = ma \quad \Rightarrow a = -0{\cdot}5g$

Using $v^2 = u^2 + 2as \quad 0 = 49 - gs$

$\Rightarrow s = \dfrac{49}{9.8} = 5$ metres

Example 4

An electron of mass 9×10^{-31} kilograms is moving at 8×10^6 ms^{-1} when it enters an electric field which produces a force of $1{\cdot}8 \times 10^{-15}$ N at right angles to its initial velocity for a period of 5×10^{-9} seconds. Find its final velocity.

Solution

Let the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ be in the directions of the initial velocity and the force respectively, then $\boldsymbol{u} = 8 \times 10^6\, \boldsymbol{i}$ and $\boldsymbol{a} = \dfrac{1{\cdot}8\times10^{-15}}{9\times10^{-31}}\boldsymbol{j} = 2 \times 10^{15}\boldsymbol{j}$.

$v = u + ta$ so $v = 8 \times 10^6 i + 2 \times 10^{15} t j$

When $t = 5 \times 10^{-9}$ $v = 8 \times 10^6 i + 2 \times 10^{15} \times 5 \times 10^{-9} j = 8 \times 10^6 i + 1 \times 10^7 j$

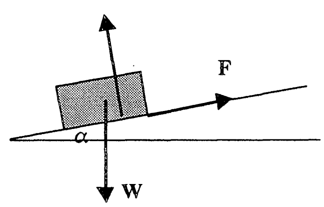
Hence velocity is $2\sqrt{41} \times 10^6$ ms^{-1} at an angle of $\tan^{-1} 1{\cdot}25$ to original direction.

Example 5

A block of mass *m* kg is sliding down a plane inclined at an angle of α to the horizontal. If the coefficient of dynamic friction between the block and the plane is μ find the acceleration of the block.

Solution

The forces acting on the block are – its weight, the normal reaction to the plane and friction, which is opposing motion, acting up the plane.



$F = \mu R$

There is no motion perpendicular to the plane so resolving perpendicular to the plane:

$R = W\cos\alpha = mg\cos\alpha$

Acceleration is down the plane, and the resultant force in this direction is:

$mg\sin\alpha - F$

Thus force causing acceleration is $mg\sin\alpha - F = mg\sin\alpha - \mu R = mg\sin\alpha - \mu mg\cos\alpha$

$F = ma$

Hence acceleration $= a = \dfrac{F}{m} = g\sin\alpha - \mu g\cos\alpha = g(\sin\alpha - \mu\cos\alpha)$ ms^{-2}

RESOURCES/EXAMPLES

S&T Chapter 3 (Force and Newton's Laws)

Pages 42,43: Ex 3A, N^{os} 3 – 12, 14 – 23 Pages 46,47; Ex 3B

Chapter 5 (Equilibrium and Acceleration under Constant Forces)

Page 98; Ex 5C, N^{os} 1 – 5, 12, 13 Page 100; Ex 5D, N^{os} 1 – 3, 10 – 12

Chapter 6 (Friction)

Page 111; Ex 6A, N^{os} 2, 5 – 11, 14 – 18

Chapter 16 (Use of Calculus)

Page 410; Ex 16C, N^{os} 1,2,12,13

RCS Chapter 3 (Force and Motion)

Page 46 – 48; Ex 3B Pages 50,51: Ex 3C N^{os} 1 – 13, 14 – 16

Chapter 13 (Variable Acceleration)

Pages 249 – 259; Ex 13A, N^{os} 1,2,11,12

Chapter 4 (Resolving)

Page 60; Ex 4A, N^{os} 2 – 7 Page 61; Ex 4B N^{os} 1,5,7,9

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Pages 86,87; Ex 5C, N^{os} 1 – 3

TG Chapter 6 (Newton's First and Second Laws)

Pages 80 – 82; Ex 6.2A, N^{os} 1–9:Page 84; Ex 6.2B N^{os} 1–5:C^{ons} Ex N^{os} 1-3

Chapter 7 (Motion with Variable Force and Acceleration)

Page 106, Consolidation Ex, N^{os} 2,6

B&C Chapter 5 (Newton's Laws of Motion)

Pages 151-156; Ex5b, N^{os} 1 – 3,5,6,8 – 13: Page 176, Misc Ex 5, N^{os} 1,4

OG Chapter 8 (Forces on a Particle)

Pages 425,426; Ex 8.1:2, N^{os} 1 – 3: Page 437, Ex 8.2:1, N^{o} 4

Pages 448,449; Ex 8.3:1, N^{os} 1 – 10: Page 452; Ex 8.3:2, N^{os} 1 – 11

Page 455; Ex 8.3:3, N^{os} 1 – 10, 21