RECURRENCE RELATIONS

EXERCISE A

1 A certain population of bacteria increases at the rate of 15% per hour.
If the initial population is 120 what would it be 3 hours later ?
How long would it take to double in number ?

2 A machine costing £9 600 initially depreciates each year by 16% of its value at the end of the previous year . Find its value at the end of the third year.

3 At the beginning of each year a lady placed £1 000 in an account paying interest at the rate of 12% per annum.
What was her total investment worth immediately after her fifth payment ?

4 A gardener prunes 50cm off a hedge each spring to keep it healthy.
It then increases its height by 20% each summer.
If it is 3metres high after its first pruning, how tall should it be after its fifth ?

EXERCISE B

1 Under laboratory conditions a particular type of protozoa is known to grow at the rate of 25% per day. If the initial protozoa population is 100, how long will it take to double in size ?
What is the size of the population after seven days ?

2 At the end of 1974 the population of India was 586 million and the net annual increase in population (ie births + immigration - deaths - emigration) was 2%.
If this rate had remained the same what would the population have been at the end of 1984 ?

3 In a marathon dancing competition it is reckoned that 8% of the couples on the floor will drop out each hour.
If 250 couples were on the floor at the start of the competition, about how many can be expected to remain after :-
(a) 3 hours
(b) 10 hours
(c) 24 hours ?

EXERCISE B
(CONTINUED)

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4 Light filters are made from translucent plastic which has the property that a sheet 1mm thick reduces the intensity of light passing through it by 10%.
How many such sheets would be necessary to reduce the intensity of a beam of light to about 50% of its original intensity ?

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5 On your sixteenth birthday and on every birthday thereafter you receive £60 which you put into a bank account which pays interest at 7% per year.
How much money will you have in this account on your 21st birthday ?

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6 You borrow £6 000 to buy a car and agree to make monthly payments of £150.
Interest is calculated annually at 12% on the outstanding balance at the start of each year.
Establish a recurrence relation to describe your position at the end of each year.
During which year will you finally pay off the money you borrowed ?

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7 In Codland the Minister of Fisheries is being pressed by fishermen to control the population of seals in the waters off the coast of Codland.
He instructs his scientists to carry out a survey as a result of which they report

"Our best estimate is that the population is growing by 10% each year but it could be anything from 8% to 12%. The population at the beginning of 1986 was about 1 000 seals but here again we cannot be certain - it could be anything from 800 to 1 200"

The Minister, a "mean" mathematician, is anxious to placate his fishermen and introduces a policy of culling 100 seals each year.
Write a report on the possible effects of this policy in 5 years time.

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8 Biologists calculate that, when the total amount of a certain chemical in a sea loch reaches 5 tonnes, the water in the loch is sufficiently polluted to endanger the life of fish.
The natural flushing action of the loch is known to remove 40% of the chemicals each week.
A factory which produces this chemical as a by-product seeks permission from the Local Water Authority to release it into the loch in batches of 2 tonnes each week.
Should the Authority give its permission ?

RECURRENCE RELATIONS EXERCISE C

1 The Christmas tree in a Dundee Square has 260 light bulbs, 15% of which fail in any one year. If it is decided to replace 30 failed bulbs each year can it be guaranteed that at least 65% of the bulbs will always be lit ?

2 It is reckoned that the drug Drazepam has a decay factor of 0.7. (ie 70% of the drug will remain in the bloodstream 24 hours after a dose is administered)
A new patient is prescribed this drug and takes 10mg of the drug at the same time each night.

(a) Explain why the level of Drazepam in the patient's bloodstream can be modelled by the recurrence relation

$$d_{n+1} = 0.7d_n + 10 \ ; \ d_0 = 0$$

where d_n is the number of mg of Drazepam in the blodstream after n nights.

(b) What will happen if the patient takes the drug over a long period of time ?

3 The gardens in Princes Street in Edinburgh are combed once per week for litter and as a result 80% of the litter is removed.
Each week 200kg of new litter is dropped.
What happens to the mass of litter in the gardens given that the initial mass of litter was 100kg ?

4 The deer population of an uninhabited island is 500 on the 1st January of a certain year. The "natural rate of increase" in the deer population is estimated to be 8% per year, all calves being born in the spring.
The owner of the island decides to let hunters kill 50 deer each autumn, but states that, if the population falls below 400 deer, he will terminate the hunting until the population of deer again goes over 500.

(a) After how many years will the deer population fall below 400 ?
(b) After the hunting is stopped how many years will pass before the deer population is again over 500 ?
(c) If the owner of the island decides to allow 50 deer to be killed each autumn for an indefinite period without stopping, what will happen ?

5 A car radiator holds 8 pints of water when full.
Each month 15% of the water is lost and the owner adds 1 pint of water to top up the radiator.
If the level of the water goes below 6 pints it is dangerous to drive the car.
Is the owner's policy safe ?

6 At the beginning of a certain year in World War II Bomber Command had 10 000 bombers for use in raids against the enemy.
The Commander-in-Chief made the following estimates :-

(i) Each month 20% of the bombers will be lost in action
(ii) 1 200 new bombers will be delivered at the end of each month.

(a) How many bombers will be in service at the end of the year ?
(b) If the war continued indefinitely, what would the long-term position be ?

7 An antibiotic in the body decays by a factor of 0.92 during each 10-minute period after it is administered. (ie 92% of drug remains in the bloodstream during each 10 minute period)

(a) If an initial dose of 100 units is administered how much will remain in the bloodstream after 1 hour ?

(b) A doctor prescribes this antibiotic for a patient.
It is known that, to be effective, a level of 100 units must be maintained and that the antibiotic becomes dangerous if the level exceeds 300 units.
The doctor prescribes 100 units per hour.
Is this satisfactory ?

8 In January 1980 a fish farmer released 100 000 trout into a small loch.
The natural rate of increase in the fish population is 25% per year with fish being born in the spring.
Each summer the farmer lets anglers catch a total of 5 000 trout from the loch.

(a) How many fish will be in the loch 5 years later ?
(b) If this situation is allowed to continue indefinitely what would happen to the fish population ?

Recurrence Relations Solutions

EXERCISE A

1. $A_0 = 120$; $A_{n+1} = 1.15A_n$; $A_n = 1.15^n.A_0$;

   $A_3 = 1.15^3 . 120 = 183$; $A_4 = 210$; $A_5 = 242$

   Doubled after 5 years

2. $A_0 = 9600$; $A_{n+1} = 0.84A_n$; $A_n = 0.84^n.A_0$;

   $A_3 = 0.84^3 . 9600 = £5689.96$

3. $A_0 = 1000$; $A_{n+1} = 1.12.A_n + 1000$

   Using calculator . . . 1000, 2120, 3374.40, 4779.33

4. $A_0 = 300$ {height after first pruning}

   $A_1 = (A_0 + 20\% \text{ of } A_0) - 50$

   $A_{n+1} = 1.20A_n - 50$

   Using calculator . . . 300, 310, 322, 336.4, 353.7, . . .

EXERCISE B

1. protozoa $A_0 = 100$; $A_{n+1} = 1.25A_n$; $A_n = 1.25^nA_0$

   A = 100, 125, 156, 195, 244 {to nearest whole number}

   $2A_0 = 1.25^nA_0 \Rightarrow 1.25^n = 2 \Rightarrow n \ln 1.25 = \ln 2$

   $\Rightarrow n = \ln 2 / \ln 1.25 = 3.106$

   Just over 3 days to double the population.

   $A_7 = 1.25^7 . 100 = 477$ {to nearest whole number}

2. $A_0 = 586$ M ; $A_{n+1} = 1.02A_n$; $A_n = 1.02^nA_0$

   $A_{10} = 1.02^{10} . 586 = 714$ M {to the nearest million}

3. $A_0 = 250$; $A_1 = A_0 - 8\% \text{ of } A_0 = 0.92A_0$; $A_{n+1} = 0.92A_n$; $A_n = 0.92^nA_0$

   a) $A_3 = 0.92^3 . 250 = 195$ b) $A_{10} = 0.92^{10} . 250 = 109$

   c) $A_{24} = 0.92^{24} . 250 = 34$

4. $A_0 = ?$; $A_1 = A_0 - 10\% \text{ of } A_0 = 0.9A_0$; $A_{n+1} = 0.9A_n$; $A_n = 0.9^nA_0$

   Find a few values : $A_2 = 0.9^2A_0 = 0.81A_0$; $A_5 = 0.9^5A_0 = 0.59A_0$;

   $A_7 = 0.9^7A_0 = 0.478A_0$ A_7 is about 1/2 of A_0 and so we need 7 screens to reduce the intensity below 50% of the original value.

   {$0.5A_0 = 0.9^nA_0 \Rightarrow n \ln 0.9 = \ln 0.5 \Rightarrow n = 6.58 \Rightarrow$ 7 screens needed}

5. $A_0 = 60$; $A_1 = (A_0 + 7\% \text{ of } A_0) + 60 = 1.07A_0 + 60$; $A_{n+1} = 1.07A_n + 60$

   Use calculator to get 124.20, 192.89, 266.40, 345.04, 429.20

£429.20 on 21st birthday

6. $A_0 = 6000$; $A_1 = (A_0 - 1800) + 12\%$ of $(A_0 - 1800) = 1.12(A_0 - 1800)$

   $A_{n+1} = 1.12(A_n - 1800) = 1.12A_n - 2016$

   Using this relation gives £6000, £4704, £3252.48, £1626.78, account closed.

7. Codland. Extreme cases are a) 1200 seals, increasing at 12% pa

   b) 800 seals, increasing at 8%

   The actual state will be between these extreme cases.

   a) $A_0 = 1200$; $A_1 = (A_0 + 12\%$ of $A_0) - 100$; $A_{n+1} = 1.08A_n - 100$

   gives the sequence 1200, 1244, 1293, 1348, 1410, 1480, . . .

   b) $A_0 = 800$; $A_1 = (A_0 + 8\%$ of $A_0) - 100$; $A_{n+1} = 1.12A_n - 100$

   gives the sequence 800, 764, 725, 683, 638, 589, . . .

   In 5 years time the population could range from 1480 and increasing to 589 and decreasing.

8. $A_0 = 2$; $A_{n+1} = 0.6A_n + 2$ (1) sequence 2, 3 1832, 3.92, 4.35, 4.61, . . .

   Let L be the limit of (1) L = 0.6L + 2 => L = 2 / 0.4 = 5 tons

   The Authority should not give permission.

EXERCISE C

1. $A_0 = 260$; $A_{n+1} = 0.85A_n + 30$ (1) gives 251, 243, 236, 231, 226, . . .

   Let L be the limit of (1) L = 0.85L + 30 => 0.15L = 30 => L = 200

   65% of 260 = 169 and so at least 65% can be guaranteed.

2. $A_0 = 0$; $A_1 = (A_0 - 30\%$ of $A_0) + 10 = 0.7A_0 + 10$

   $A_{n+1} = 0.7A_n + 10$ (1) gives 1, 10, 17, 21.9, 25.3, 27.7, . . .

   Let L be the limit of (1) L = 0.7L + 10 => L = 33.3333 . . .

3. $A_0 = 100$; $A_1 = (A_0 - 80\%$ of $A) + 200 = 0.2A_0 + 200$ (1)

   $A_{n+1} = 0.2A_n + 200$ gives 100, 220, 244, 248.8, . . .

   Let L be the limit of (1) L = 0.2L + 200 => 0.8L = 200 => L = 250

4. $A_0 = 500$; $A_1 = (A_0 + 8\%$ of $A_0) - 50$;

   a) $A_{n+1} = 1.08A_n - 50$ gives

   500 1) 490 2) 479 3) 468 4) 455 5) 441 5) 427 7) 411 8) 394

   Population falls below 400 after 8 years.

   b) Hunting is stopped after 8 years at 394

   $A_0 = 394$; $A_1 = A_0 + 8\%$of $A_0 = 1.08A_0$

   $A_{n+1} = 1.08^n A_n$ gives 394 1) 426 2) 460 3) 496 4) 536

After another 4 years, the population rises above 500

c) $A_0 = 536$; $A_1 = A_0 + 8\%$ of $A_0 - 50 = 1.08A_0 - 50$

$A_{n+1} = 1.08^n A_n - 50$ gives 536, 529, 521, 513, 504, 494, . . .

population keeps falling!

5. $A_0 = 8$; $A_1 = (A_0 - 15\%$ of $A_0) + 1 = 0.85A_0 + 1$; $A_{n+1} = 0.85A_n + 1$ (1)

   Let L be the limit of (1) $L = 0.85L + 1 \Rightarrow 0.15L = 1 \Rightarrow L = 1 / 0.15 = 6.67$

   The owner

's policy is safe.

6. $A_0 = 10000$; $A_1 = (A_0 - 20\%$ of $A_0) + 1200 = 0.8A_0 + 1200$

   $A_{n+1} = 0.8A_n + 1200$ (1) gives 10000, 9200, 8560, . . .

   Let L be the limit of (1) $\Rightarrow L = 0.8L + 1200$

   $\Rightarrow 0.2L = 1200 \Rightarrow L = 6000$ planes

7. $A_0 = 100$; ${}_1A = 0.92A_0$ {${}_1A$ = amount after 1 ten minute period}

   ${}_6A = 0.92^6 A_0$ {${}_6A$ = amount after 6 ten minute periods}

   $A_1 = 0.92^6 A_0 + 100$ {A_1 = amount after 1 hour = ${}_6A$}

   $= 0.606A_0 + 100$ gives 100, 160.6, 197.3, 219.6, 233.1, . . .

   Let L be the limit of the sequence $L = 0.606L + 100$

   $\Rightarrow 0.394L = 100 \Rightarrow L = 253.8$ Amount prescribed is satisfactory.

8. $A_0 = 100000$; $A_1 = (A_0 + 25\%$ of $A_0) - 5000 = 1.25A_0 - 500$

   $A_{n+1} = 1.25A_n - 500$ gives 100000, 120000, 145000, 176250, 215313, . . .

   The numbers increase until the loch cannot cope with the numbers.